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# ELEMENTS OF GRAPHIC DYNAMICS

## AN ELEMENTARY TEXT-BOOK FOR STUDENTS OF MECHANICS AND ENGINEERING

BY

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*WITH NUMEROUS ILLUSTRATIONS AND WORKED EXAMPLES*

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# PREFACE

GRAPHIC methods have been successfully applied for very many years for the solution of mechanical problems, particularly those arising in engineering, and nearly all students of mechanics prefer the graphic methods to those based upon mathematical analysis. The reason is that argument based upon algebraic symbols does not give to most minds such a clear conception as that based upon diagrams; we believe that in all walks of life the diagram method of appeal to the mind (*i.e.* by pictures) is the most effective.

The champions of the mathematical method of argument have in the past laid great stress upon its value in the development of reasoning powers, suggesting that such reasoning is absent in the graphic method. Although this may be true in cases in which graphic methods are taught merely as mechanical processes, it is certainly not necessarily true.

The application of graphic methods to the problems arising in structural engineering has been very fully developed under the name of "Graphic Statics," but comparatively little attempt appears to be made in our schools and universities to develop the dynamic problems arising in mechanical engineering with the same logical sequence, the graphic constructions employed usually being introduced in a rather random manner.

In the present book, which we call *Elements of Graphic Dynamics,* we have attempted to present the laws of elementary dynamics with particular reference to engineering

problems in a manner analogous to that employed in Graphic Statics. We wish to lay particular stress upon the fact that by studying the subject from the graphic standpoint we are not forced to employ in our calculations graphic methods when numerical substitution in a formula will be more convenient; we show that for certain special cases formulæ can be derived—and we would remind our readers that it is only in special cases that formulæ can be derived—by the ordinary methods of mathematical analysis. We do not say that graphic methods are better than mathematical analysis; we do say that most engineering students will learn more from graphic methods than from the other.

A considerable portion of the subject-matter of this book has already appeared in article form in *Machinery*, the *Engineering Review*, and the *Mechanical World*, and the author has to thank the editors of the two first-mentioned journals for the use of blocks.

EWART S. ANDREWS.

204 6 Bank Chambers,
329 High Holborn, W.C. 1.
*March*, 1919.

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# ELEMENTS OF
# GRAPHIC DYNAMICS

## CHAPTER I

### THE SUM CURVE AND ITS APPLICATIONS TO WORK CURVES

**The Sum Curve or Graphical Integration.**—A number of problems in machine design involve the operation known to mathematicians as "integration," which is more simply described from the graphical standpoint (which is much clearer to most engineers) as "the sum curve." We will first describe the sum curve construction, and next take its application to problems involved in machine design.

DEFINITION.—Suppose that there are two curves X and Y, Fig. 1, which are so related that the ordinate $a\,b$ of the curve Y at any point gives to some scale the area of the curve A up to the same point; then Y is called the sum curve of X, which is usually called the primitive curve.

We can show as follows that the ordinate of the curve X at any point represents the slope of the curve Y at the corresponding point.

Let $a\,b$ and $d\,f$ be two ordinates of the sum curve taken very close together, so close, in fact, that by joining $a\,d$ and producing we get a tangent to the sum curve (for clearness on the diagram they are shown fairly wide apart). The slope of this tangent is a measure of the slope of the curve at the given point.

Let $a\,g$ be drawn perpendicular to $d\,f$,

$$\text{then } \tan\theta = \text{slope of sum curve} = \frac{d\,g}{a\,g} \quad \ldots\ldots (1)$$

Now by the definition of sum curve, the ordinate $a\,b$ is equal to the area of the original curve A up to $b$, and $d\,f$ is the area up to $f$.

$$\begin{aligned} \therefore\ df - ab &= \text{area of curve up to } f - \text{area of curve up to } b \\ &= \text{area of strip } b\,c\,e\,f \\ &= x \times bf. \end{aligned}$$

But $df - a\,b = d\,g$

$$\therefore\ \tan\theta = \frac{d\,g}{a\,g} = \frac{x \times bf}{a\,g} = x$$

$$\therefore\ x = \text{ordinate of curve A} = \text{slope of sum curve B.}$$

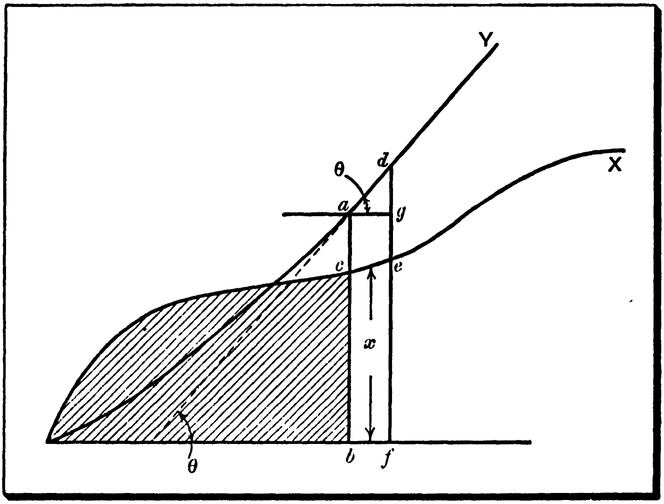


FIG. 1.—The Sum Curve.

**Construction of Sum Curve.**—Let A C D, Fig. 2, be any primitive curve or a straight base A B. Divide A B into any number of parts, not necessarily equal (but for convenience of working they are generally all taken as equal, except the last, when the base will not divide up exactly). These

so-called base elements should be taken so small that the portion of the curve above them may be taken as a straight line. About 1 cm., or 0·4 in., will usually be a suitable size, and in most cases a smaller element 11 will come at the end. Find the mid-points, 1, 2, 3, etc., of each of the base elements, and let the verticals through these mid-points meet the curve in 1*a*, 2*a*, 3*a*, etc. Now project the points on to a vertical line A E, thus obtaining the points 1*b*, 2*b*, 3*b*, etc.,

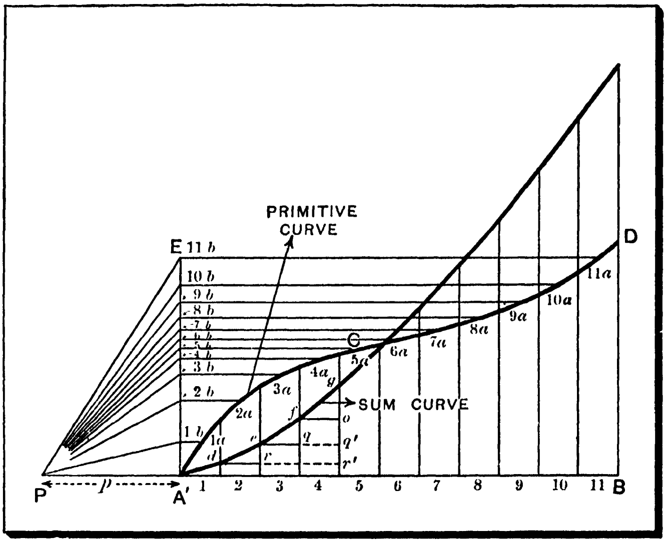


FIG. 2.—Graphical Construction of the Sum Curve.

and join such points to a pole P on A B produced and at some convenient distance *p* from A. Across space 1 then draw A *d* parallel to P 1*b*, *d e* across space 2 parallel to P 2*b*, and so on, until space 11 is reached. Then the curve A *d e* . . . is the sum curve of the given curve, and to some scale the ordinate at B represents the area of the whole curve.

PROOF.—Consider one of the elements, say 4, and draw *f o* horizontally.

Now $\Delta\, f, g, o$ is similar to the $\Delta$ P, 4*b*, A

$$\therefore \frac{go}{fo} = \frac{4b, \mathrm{A}}{\mathrm{PA}}$$

but $\mathrm{PA} = p$ and $4b, \mathrm{A} = 4, 4a$

$$\therefore go = \frac{fo \times 4, 4a}{p} = \frac{\text{area of element 4 of curve}}{p}$$

$$\text{Similarly, } fq = \frac{\text{area of element 3 of curve}}{p}, \text{ and so on.}$$

$$\therefore \text{ Ordinate through } g = go + fq + \ldots$$

$$= \frac{\text{area of first four elements of curve}}{p}$$

The curve A *d e* . . . *h* is the sum curve required.

Then if the sum curve ordinate be measured on the vertical scale, and $p$ be measured on the horizontal scale, the corresponding area of the curve will be equal to $p \times$ ordinate of sum curve.

It is obviously advisable to make $p$ some convenient round number of units.

The sum curve obtained by this method may have the same operation performed on it, and thus the second sum curve of the primitive curve is obtained, and so on.

*If the operation be performed on a rectangle, the sum curve will obviously come a sloping straight line; and if the sum curve of a sloping straight line be drawn, it will be found to be a parabola.*

In the case in which it is required to apply this construction to a curve which is not on a straight base, the curve is first brought to a straight base as follows :

Suppose A *c* B *d*, Fig. 3, is a closed curve. Draw verticals through A B to meet a horizontal base A′B′. Divide the curve into a number of segments by vertical lines at short distances apart, and set up from the base A B lengths $a_1$, $b_1$, etc., equal to the vertical portions *a*, *b*, etc., on the curve. Joining up the points thus obtained, we get the corresponding curve $\mathrm{A}'c_1\mathrm{B}'$, on a straight base. Before leaving the general consideration of sum curves, in order to consider its applica-

tion to dynamical problems, we will just note that the *form* of the sum curve will be the same if we started it at some point upon the line A B instead of exactly at the point A; the curve would only be shifted bodily downwards or upwards relatively to the base line A B. It also follows from a consideration of the construction that the slope of a curve at any point is quite independent of this initial ordinate, *i.e.* that the position of the base line of the curve makes no difference at all to the slope of the curve. The initial ordinate of a sum curve in any given application of the construction

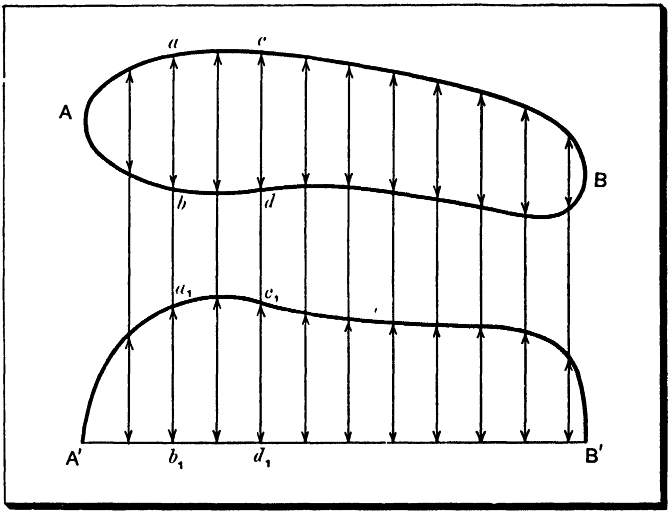


FIG. 3.

is usually governed by conditions specifically laid down by the nature of the problem; in the case, for instance, in which the sum curve is a curve of velocities obtained from a consideration of the resultant effort acting upon the body in the manner to be next explained, the initial ordinate will be determined by the velocity which the body has at the commencement of the period under consideration.

**Effort and Work Curves.**—Suppose that the force acting upon a body in the direction of its motion and driving

it forward varies so that when we plot a diagram of the force at various points we get a curve A B C, Fig. 4, which is usually called the *effort curve.* Consider two points B and C which are so close together that the resultant effort or force E may be regarded as constant over the length. Then the work done over this length will be equal to constant force $\times$ distance moved $=$ E $\times$ $x$ $=$ area of the shaded strip of the curve. The reader will see that the smaller we make the distance B C the more nearly true will be the statement that the area of the strip is equal to E $\times$ $x$, but that for

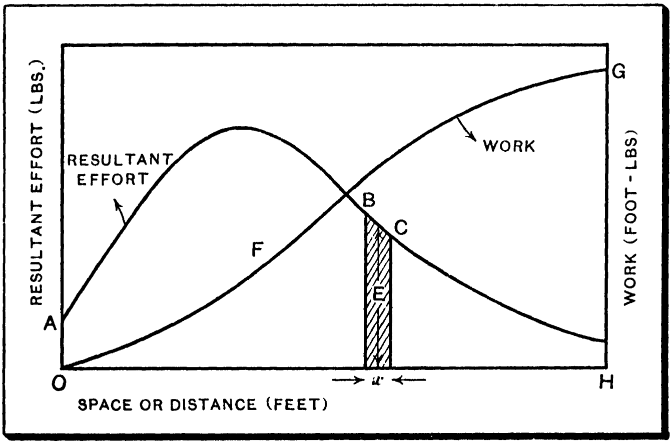


FIG. 4.—Effort and Work Curves.

comparatively long lengths the statement is only approximate. If now we consider the whole base O H to be divided up into short lengths, the same argument will hold for each strip of the curve, so that, adding together these separate strips, we see that the total work done in moving the body from O to H is represented by the area O A B C H.

Or we get the rule that :

*The work done is represented by the area beneath the effort curve.*

Now suppose that we wish to find the work done up to

various points along the base and to obtain a diagram representing to some other scale the work done. Such a diagram will be of the form shown by the curve O F G in the figure and is called the *work curve*. Consider any point on this curve. Then the ordinate represents the work done in moving from O to the point, and this is also given, as we have proved above, by the area below the effort curve. We see, therefore, that the ordinate of the work curve at any point represents the area of the effort curve up to the same point; therefore *the work curve is the sum curve of the effort curve.*

SCALES.—In all graphical constructions the principal difficulty that is experienced is that of the determination of the scales; it is essential that we should be quite clear about these scales before we proceed further. We will first explain them in general terms and then give a numerical example. Let the effort or force scale be $1'' = x$ lb. Let the space or distance scale be $1'' = y$ ft. Let the polar distance be $p$ actual inches. Then the work scale will be $1'' = p\,x\,y$ ft.-lb. If, for instance, the space scale were $1'' = \frac{1}{2}$ ft. (6 in.), and the effort scale $1'' = 400$ lb., and $p = 3''$, the work scale would be $1'' = 3 \times \frac{1}{2} \times 400 = 600$ ft.-lb. This is not a very convenient scale, and could be improved upon greatly by altering $p$ so as to make it $1'' = 500$ ft.-lb.; then $p$ would be equal to $\frac{500}{\frac{1}{2} \times 400} = 2\frac{1}{2}$ in.

***Wherever possible, the scales should be chosen so that results can be read off with practically no mental arithmetic to perform.***

**Special Cases.**—(1) *Constant Effort.*—It is clear, from the statement on p. 4, that, if the effort is constant and the effort diagram is therefore a rectangle, the work curve will be a sloping straight line. If, therefore, this constant effort is E and the space or distance through which the body moves is $s$, we shall have

$$\text{Work done} = \text{area of rectangle} = \text{E}\,s.$$

(2) *Regularly Increasing or Decreasing Effort.*—If the effort increases or decreases uniformly from the commencement of

the movement, the effort curve will be a sloping straight line, and since the sum curve of a sloping straight line is a parabola, the work curve will be a parabola. In the case of increasing effort the effort curve will slope upwards and the slope of the parabola will increase; in the case of decreasing effort the effort curve will slope downwards and the slope of the parabola will decrease.

If the effort is zero at the beginning of the movement and is E after a distance $s$, we shall have :

Work done

$$= \text{area of triangle of base } s \text{ and height E} = \tfrac{1}{2}\,\text{E}\,s \quad \ldots\ldots \quad (2)$$

NUMERICAL EXAMPLE.—*The force urging a body forward increases uniformly from zero to 2,000 lb. during the first 15 ft. of movement; it then remains constant for the next 20 ft.; and finally decreases uniformly to zero in a further 20 ft. Find the work done and the constant force which would do the same amount of work in moving the body through the same distance.*

Fig. 5 shows the effort curve in this case. Therefore the work done is given by the area of the figure A B C D.

This is equal to

| | | ft.-lb. |
|---|---|---|
| Area of $\triangle$ A B E | $= \frac{1}{2} \times 15 \times 2{,}000$ | $= 15{,}000$ |
| Area of rectangle B C F E | $= 20 \times 2{,}000$ | $= 40{,}000$ |
| Area of $\triangle$ C D F | $= \frac{1}{2} \times 20 \times 2{,}000$ | $= 20{,}000$ |
| | Total work done | $= 75{,}000$ ft.-lb. |

The total distance moved is 55 ft. If, therefore, a constant force F were acting, we should have $55\,\text{F} = 75{,}000$;

$$\therefore\ \text{F} = \frac{75{,}000}{55} = \underline{1{,}364 \text{ lb. nearly.}}$$

We show in dotted lines in Fig. 5 the work curve for this case. The portion A H is a parabola with vertex at A; H J is a straight line, and J K is a parabola with vertex at K. The student should make this construction as an exercise.

Take, for instance, as scales: Distance, $1'' = 10$ ft.; force, $1'' = 1{,}000$ lb.; polar distance, $p = 2{\cdot}5$ actual inches.

Then the work scale will be $1'' = 2{\cdot}5 \times 10 \times 1{,}000 = 25{,}000$ ft.-lb.

D K should therefore be 3 in.

**Work against Resistance.**—In every case that arises in practice there is a force resisting the movement of a body under the driving force or effort; such resisting force is called the *resistance*, or sometimes the "external resistance."

Take the case of a steamboat; the steam acting upon the pistons and thence upon the propeller causes a certain tractive

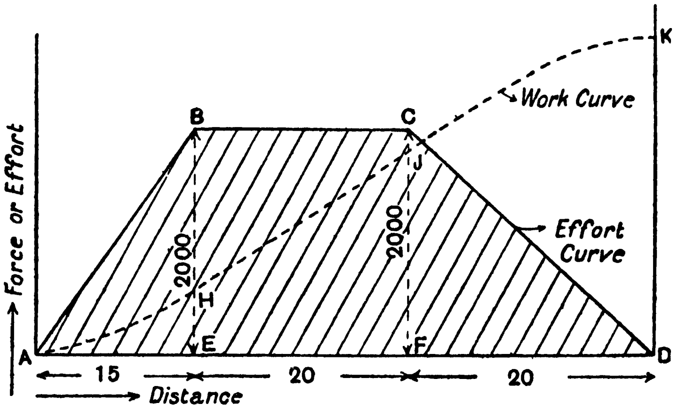


Fig. 5.

effort F to be exerted, tending to push the steamer forward; the resistance of the water and other external forces tending to resist the forward motion of the steamer cause a resistance force S to be exerted in the opposite direction. If F is greater than S at any instant, work will be done upon the steamer, and as such work cannot be lost it becomes converted into increased kinetic energy; and if F is less than S, the kinetic energy of the steamer will decrease. This is expressed in simple language by saying that, if F is greater than S, the speed of the vessel will increase, but that, if S is greater than F, the speed will decrease. The kinetic energy can be regarded as

energy stored up for use in emergency ; if the effort is less than the resistance, the body gives up some of its kinetic energy to make up the difference between the work done by the effort and the resistance. When this difference in work is equal to the whole kinetic energy that the body possessed in the first place, the body will stop moving.

We have here an example of a case in which a body moves in a direction opposite to that in which the resulting force upon it acts. When the direction of movement is opposite to that of the force we shall speak of the force as taking work from the body.

Resistances are nearly always what may be called "induced" or "passive"; that is to say, they disappear directly the body comes to rest. The resistance to the motion of a steamer increases very quickly with the speed, and we soon get to a speed which may be regarded as the most economical. A slight increase of speed over this will require more coal and cost more money than the saving in time is worth.

We have another similar example in racing motor-cars. An ordinary 12-h.p. car can do 30 miles an hour, but to get 60 miles an hour we have to increase the horse-power to something like 80 or more.

**Graphical Representation of Effort and Resistance.**— Suppose that the effort in moving a body from a point X to a point T varies in the manner indicated by the curve A B C, Fig. 6, and that the resistance varies in the manner indicated by the curve D B F. Then if we take two points K L very close together on the base—so close that the effort E and resistance S may for all practical purposes be considered as constant over the length—the work done upon the body by the effort from K to L is equal to force $\times$ distance $=$ E $\times$ K L $=$ area of strip E G L K. Therefore, as already shown, the total work done on the body by the effort in moving from X to T is equal to the area A B C T X.

Similarly, the work taken by the resistance from the body

in moving from K to L is equal to S × K L = area of strip H J L K, so that the total amount of work taken from the body in moving from X to T is equal to the area D B F T X.

Now the resultant work on the body is equal to work done by the effort — work taken away by the resistance

= area A B C T X — area D B F T X

= area A B D — area B F C.

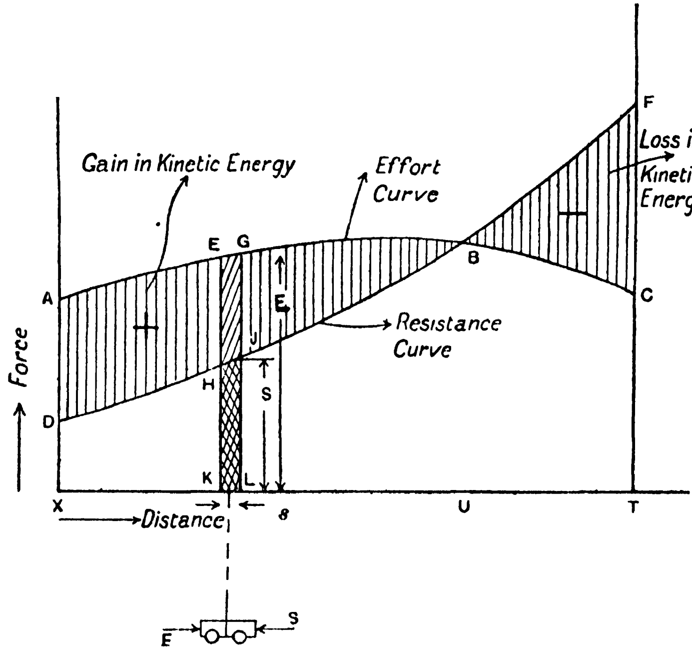


FIG. 6.—Work against Resistance.

At any intermediate point such as K the excess of work done by the effort over the work expended in overcoming the resistance is the difference between the areas X A E K and X D H K.

Therefore, between the points X and U the body increases in kinetic energy by the amount represented by the area A D B, and it then loses in going between U and T an amount of kinetic energy represented by the area B F C. We have

not yet explained how the kinetic energy can be expressed in terms of the velocity, but we have seen that the velocity is an indication of the kinetic energy; consequently at the point U the body has the maximum amount of kinetic energy, and therefore has the maximum velocity or speed.

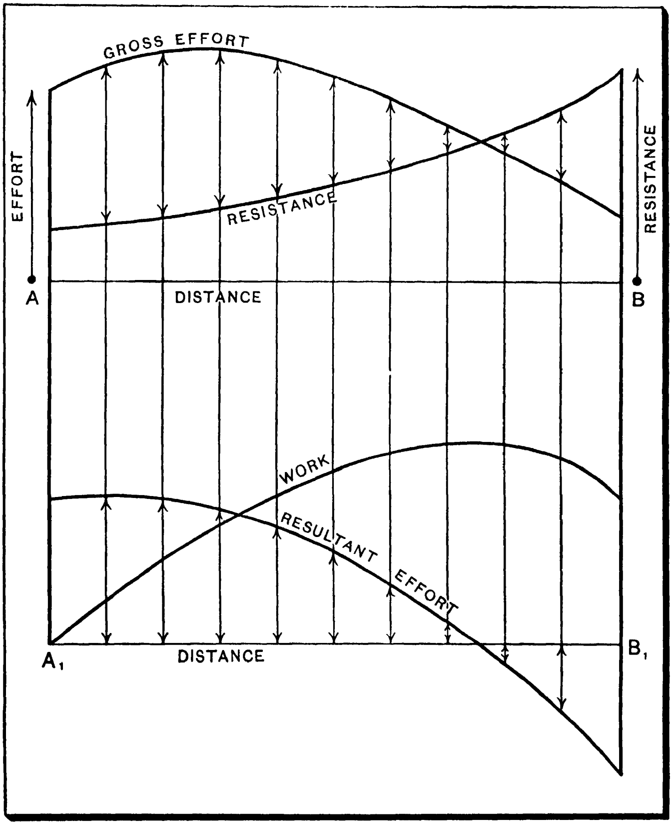


FIG. 7.—Gross Effort and Resistance Diagrams.

If the conditions were reversed so that the resistance were at first greater than the effort and less at the end, the body would be losing kinetic energy up to the point U; U would then be the point of least kinetic energy, and therefore of least velocity.

We thus see that, in order to obtain the work diagram for the case of a body moving against a resistance, we first draw the effort curve (which for distinction we may call the gross effort curve) and the resistance curve to the same scale on the same base; we then plot the difference between these two curves, giving the resultant effort curve as shown in Fig. 7, and then draw the sum curve of this curve to obtain the work curve.

It will be noted that the maximum ordinate of the work curve occurs where the resultant effort curve cuts the base, *i.e.* at the point of zero resultant effort.

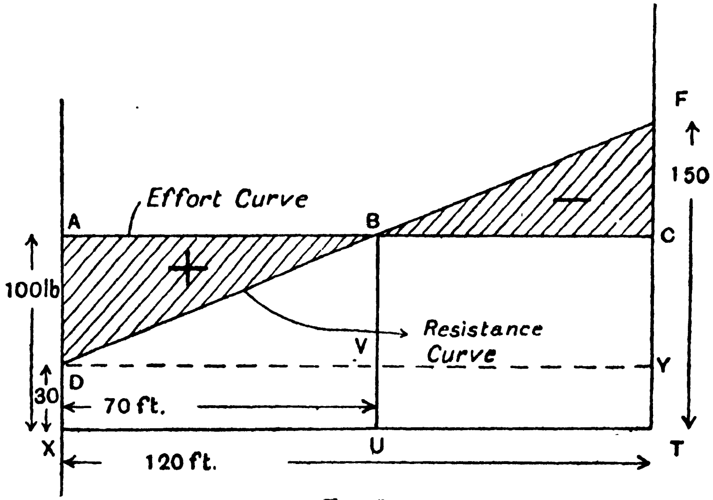


FIG. 8.

NUMERICAL EXAMPLE OF EFFORT AND RESISTANCE.—*A body is being urged forward by a constant force equal to* 100 *lb., and over a distance of* 120 *ft. the resistance increases uniformly from* 30 *lb. to* 150 *lb. At what point will the body move with the greatest velocity? How much kinetic energy will the body then have gained, and how much will it have gained at the end of the* 120 *ft.?*

Referring to Fig. 8, A B C is the effort curve and D B F the resistance curve.

The point U gives the point of maximum velocity. The distance X U can be measured by drawing the diagram to scale, *e.g.* distances to a scale 1″ = 20 ft., and forces to a scale 1″ = 50 lb.; one square inch of area would represent 20 × 50 = 1,000 ft.-lb. It will come to 70 ft.

By calculation we should proceed as follows: Draw D Y horizontally as indicated in dotted lines.

Then $\frac{DV}{DY} = \frac{BV}{FY}$, because B V is parallel to F Y.

$$\therefore DV = \frac{DY \cdot BV}{FY} = \frac{120\,(100 - 30)}{150 - 30} = \frac{120 \cdot 70}{120} = \underline{70 \text{ ft.}}$$

$$\begin{aligned}
\text{Gain in K.E. up to U} &= \text{area of } \triangle\ ABD = \tfrac{1}{2} AB \cdot AD \\
&= \tfrac{1}{2} \cdot 70 \cdot 70 \\
&= \underline{2{,}450 \text{ ft.-lb.}}
\end{aligned}$$

$$\begin{aligned}
\text{Gain in K.E. up to T} &= \text{area of } \triangle\ ABD - \text{area of } \triangle\ BFC \\
&= 2{,}450 - \tfrac{1}{2} \cdot 50 \cdot 50 \\
&= 2{,}450 - 1{,}250 \\
&= \underline{1{,}200 \text{ ft.-lb.}}
\end{aligned}$$

**Mean Effort.**—It is sometimes convenient to find the uniform effort which, acting over the same distance, will do the same amount of work as a variable one; this is called the *mean effort.* Referring to Fig. 4, let $E_m$ be the mean effort; then work done by $E_m = E_m \times OH$. But the work done by the mean effort has to be equal to the work done by the variable effort.

$$\therefore E_m \times OHJ = \text{area } OABCH = HG;$$

$$\therefore E_m = \frac{HG}{OH}.$$

Expressing this in general terms, we have

$$\text{Mean effort} = \frac{\text{area below effort curve}}{\text{length of effort curve}}.$$

**Indicated Horse-power of Engines.**—In testing steam, gas, oil, and similar engines it is usual to measure what is called the "indicated horse-power" (I.H.P.).

The *indicated horse-power* is in a sense a measure of the power input and is calculated from diagrams drawn by an instrument called the "indicator," which automatically indicates graphically as a diagram the pressure per sq. in. of the steam or gas in the engine cylinder at the various points of the stroke. This diagram in the case of a steam-engine is somewhat as indicated in Fig. 9. The total pressure

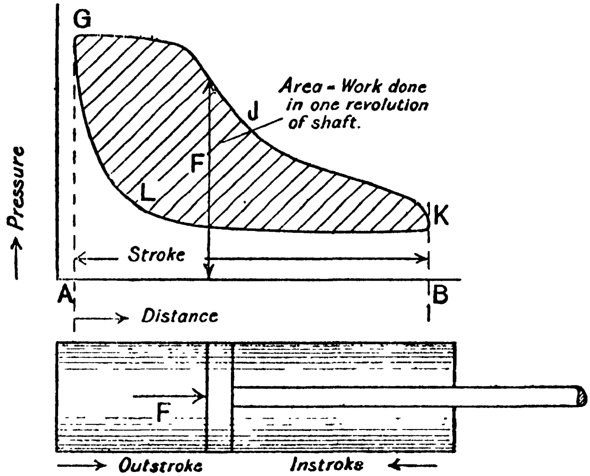


FIG. 9.

acting upon the piston is the effort, and this is equal to the pressure per sq. in. multiplied by the area of the piston, so that the indicator diagram draws for us to a reduced scale the effort curve, and we have shown already that the area under the effort curve represents the work done.

On the outstroke of the piston the work done is represented by the area A G J K B, and on the instroke the work taken from the piston in bringing it back is represented by the area B K L G A; the difference between these two areas—that is, the area shaded—represents, therefore, the work done by the steam or gas upon the piston in one double-stroke of the latter, *i.e.* in one revolution of the engine-shaft.

Therefore the mean height of this diagram, *i.e.* $\dfrac{\text{shaded area}}{\text{BA}}$, represents the mean effort. The indicator is calibrated so that, by multiplying the mean height of the diagram in inches by a constant, we get at once the mean pressure acting on the piston in lb. per sq. in. Let this mean pressure be $p_m$ lb. per sq. in.

Now let A be the area of the piston in square inches, L the length of the stroke in feet, and N the number of revolutions per min. of the engine-shaft. N is measured during the test by a counter.

Then $E_m = p_m A$.

$\therefore$ Work done per revolution $= E_m \cdot L = p_m A L$.

$\therefore$ Work done per minute $= p_m A L N$.

$\therefore$ Indicated horse-power $= \dfrac{p_m A L N}{33{,}000} \ldots$

# CHAPTER II

## SPACE, VELOCITY, AND ACCELERATION CURVES

WE will now consider the curves obtained by plotting space on distance, velocity, and acceleration upon a base representing time.

Suppose that the times are recorded at which a moving

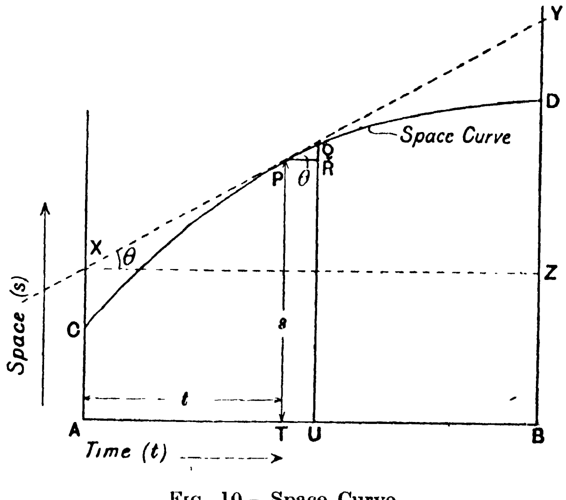


FIG. 10.—Space Curve.

point passes certain stations, and that the distances of these stations from a suitable starting point are plotted against the times of passing. The curve C P Q D, Fig. 10, obtained by joining up the points, is called the *space curve*.

At the instant from which the time is reckoned the distance

from the starting point is A C; then at any point such as P, after a time $t = AT$ has elapsed, the moving point is at a distance $s = PT$ from its starting point.

Now consider a point Q on the space curve very near to P, and let P R be drawn perpendicular to Q U; while the point has moved a distance equal to $QU - PT = QR$, the time has increased by an amount $TU = PR$.

$$\text{Now} \qquad \frac{QR}{PR} = \frac{\text{distance moved}}{\text{time taken}} = \tan\theta.$$

Next suppose that the points Q and P move closer and closer to each other; the line P Q then gradually approaches the position of the tangent X Y shown in dotted lines, and the slope of this tangent may be taken as $\tan\theta$ if P Q is sufficiently small.

Now we define the velocity at any point as the value which the $\dfrac{\text{distance moved}}{\text{time taken}}$ approaches as the distance moved becomes smaller and smaller. It follows from this that *the slope of the tangent to the space curve at any point measures the velocity at that point.*

In working from the diagram we must be careful to allow properly for the scales; referring to the figure, we have

$$\text{Velocity at given point} = \tan\theta = \frac{\text{Y Z on space scale}}{\text{X Z on time scale}}.$$

NUMERICAL EXAMPLE.—*The following results were obtained in timing a man walking over a certain distance. Find the velocity at the commencement and after* 40 *mins. from the start of the test. Find also the average velocity over the whole test.*

| Time in seconds ... | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| Distance in feet from starting point ... | 150 | 196 | 263 | 345 | 440 | 505 | 550 |

The results are plotted in Fig. 11, where B C D is the space curve and we are required to find the velocities at B

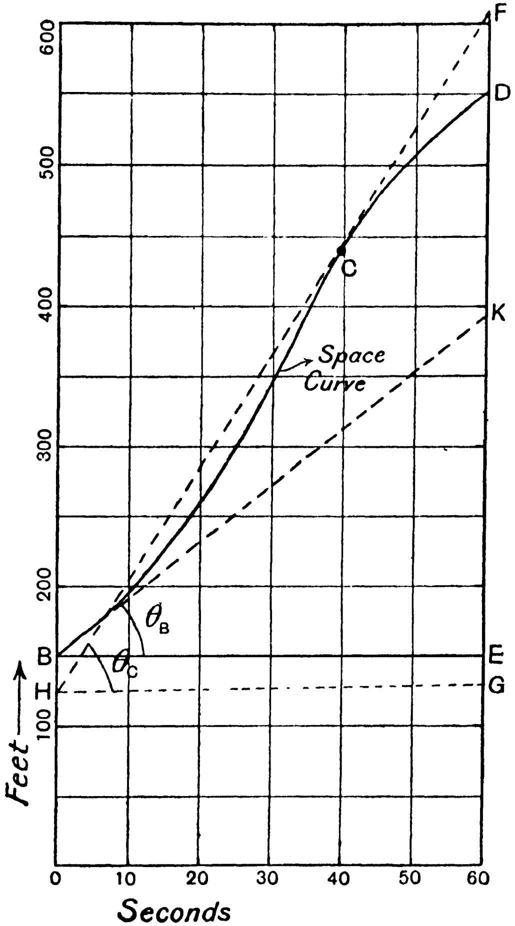


FIG. 11.

and C. To do this we have, as previously explained, to draw tangents to the space curve at B and C. This is not easy to do accurately; so far no graphical construction is known which is really more accurate than drawing a line by eye to touch the curve.

$$\text{Then velocity at B} = \tan\theta_B = \frac{\text{E K}}{\text{B E}} = \frac{241\text{ ft.}}{60\text{ secs.}}$$
$$= 4{\cdot}01\text{ ft. per sec.}$$

It is common to measure some velocities in miles per hour. Now 1 mile per hour = 5,280 ft. in 3,600 secs.

$$= \frac{5{,}280}{3{,}600} = \frac{22}{15} = 1{\cdot}467\text{ ft. per sec.}$$

$$\therefore \text{Velocity at B in miles per hour} = \frac{4{\cdot}01}{1{\cdot}467} = 2{\cdot}73\text{ miles per hour.}$$

$$\text{Velocity at C} = \tan\theta_C = \frac{\text{G F}}{\text{H G}} = \frac{480}{60} = 8\text{ ft. per sec.}$$
$$= \frac{8}{1{\cdot}467} = 5{\cdot}45\text{ miles per hour.}$$

$$\text{Average velocity} = \frac{\text{D E}}{\text{B E}} = \frac{400}{60} = 6{\cdot}67\text{ ft. per sec.}$$
$$= \frac{6{\cdot}67}{1{\cdot}467} = 4{\cdot}55\text{ miles per hour.}$$

A useful figure to remember is that 60 miles per hour is equal to 88 ft. per sec., or one mile per hour = $\frac{22}{15}$ ft. per sec.

**Velocity Curve and its Relation to the Space Curve.**—Next suppose that we know the velocity at every time and that we plot velocities upon a time base; then the resulting curve is called a velocity curve G H J K, Fig. 12. A G represents the velocity at the beginning of the period of time under consideration and is called the *initial velocity*, and will be given the letter $v_0$.

Now consider the relation between this curve and the space curve plotted on the same base (to save confusion it is preferable to plot one diagram above or below the other). We have already shown that the velocity $v$ at the middle of a very short time L M is equal to the slope of the tangent to the space curve at the corresponding point. Since L M is so very short, the slope of this tangent is given by

$$\tan\theta = \frac{\text{Q R}}{\text{P R}} = \frac{\text{Q R}}{\text{L M}};$$

$$\therefore v = \frac{\text{Q R}}{\text{L M}};$$

$\therefore$ QR = LM $\times$ $v$ = area of shaded strip of the velocity curve. But QR is the increase in the ordinate of the space curve, and we could show similarly for any other strip that the increase in the ordinate of the space curve represents the area of the corresponding strip of the velocity curve.

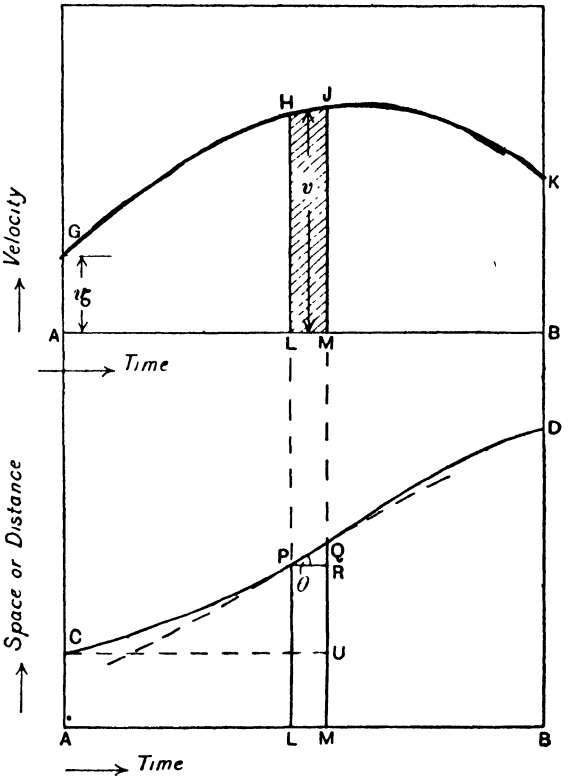


FIG. 12.—Velocity and Space Curves.

$\therefore$ QU = total increase in space from the beginning = area of velocity curve from A to M, *i.e.* area AGJM. But this is exactly the relation which we have explained between a slope curve and its primitive curve.

***Therefore the space curve is the sum curve of the velocity curve.***

We may use as a general rule the following relation, which we have proved above. If a curve A is the sum curve of a curve B, the ordinate of B at any point represents the slope of A at the corresponding point.

**Some Special Cases of Velocity and Space Curves.**—(*a*) *Constant Velocity.*—If the velocity of a point is constant, the velocity curve is a horizontal straight line, Fig. 13. If the sum curve construction be carried out for this we get a sloping straight line A D, assuming that we commence reckoning our distances from the point at which the time commences. This is because all the mid-ordinates of the velocity curve when projected horizontally come to the point G, so that all the elemental pieces of the sum curve are parallel to

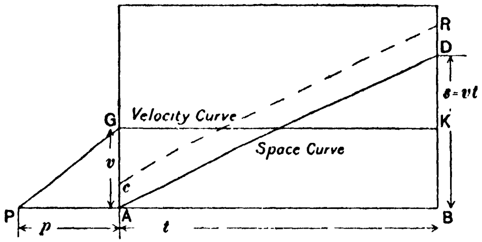


FIG. 13.—Constant Velocity.

P G. (If, instead of taking our distances from the point at which the time commences, we take them from some other point, we shall get a space curve such as $c$ R parallel to A D, but in all further cases we shall assume that the space is considered as commencing at the beginning of the time interval.)

We then have : Distance covered from A to B

$$= \text{area } A\,G\,K\,B = v\,t,$$

*i.e.* $$s = v\,t \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Since for any value of $t$ the space $s$ is equal to $v\,t$, the space curve is a straight line such that $\tan D\,A\,B = \frac{s}{t} = v.$

(*b*) *Velocity Increasing Uniformly.*—If the velocity increases

by the same amount in each unit of time, we shall obtain for our velocity curve a sloping straight line G K, Fig. 14.

If we apply to this the sum curve construction, the space curve will be found to be a *parabola* A R D.

We then have: Distance covered from A to B

$$= \text{area AGKB} = \tfrac{1}{2}\,\text{AB}\,(\text{AG} + \text{BK})$$

$$= \tfrac{1}{2}\,t\,(v_0 + v),$$

*i.e.* $$s = t\left(\frac{v_0 + v}{2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

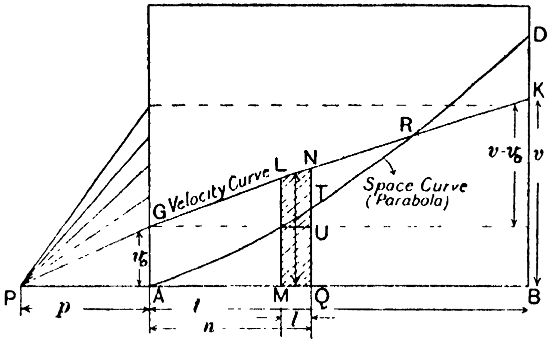


FIG. 14.—Velocity Increasing Uniformly.

We shall see later that it is sometimes more convenient to write this

$$s = \frac{t}{2}\{v_0 + (v_0 + v - v_0)\}$$

$$= \frac{t}{2}\{2v_0 + (v - v_0)\}$$

$$= v_0 t + \frac{(v - v_0)\,t}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

(*c*) *Velocity Decreasing Uniformly.*—In this case the velocity curve G K, Fig. 15, will also be a straight line, but will slope downwards. The space curve will also be a parabola A P D, but it will curve the opposite way from the previous case.

After a time *t*, therefore, we get

$$s = \text{area } AGKB$$
$$= \frac{AB}{2}(AG + BK)$$
$$= \frac{t}{2}(v_0 + v) \text{ as before}$$
$$= \frac{t}{2}\{v_0 + v_0 - (v_0 - v)\}$$
$$= \frac{t}{2}\{2v_0 - (v_0 - v)\}$$
$$= v_0 t - \frac{(v_0 - v)}{2} t \quad \ldots\ldots\ldots\ldots\ldots (4)$$

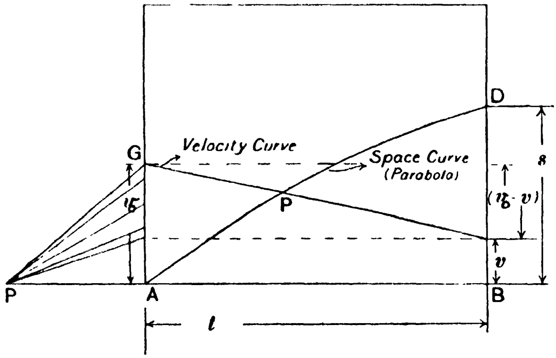


FIG. 15.—Velocity Decreasing Uniformly.

NUMERICAL EXAMPLE.—*A point starts from rest and increases its velocity uniformly for* 10 *secs., at the end of which it has a velocity of* 10 *ft. per sec. It continues to move for* 10 *more seconds at this velocity, and the velocity then diminishes uniformly for* 5 *secs., when it comes to rest. How far has it travelled?*

The velocity curve for this case is as shown in Fig. 16. For the first 10 secs. between A and C it is a sloping straight line A C; for the next 10 secs. the velocity is constant, so that the velocity curve is a horizontal straight line C D; and for the next 5 secs. the velocity falls uniformly to zero, so that the velocity diagram is the sloping straight line D B.

Now the total space covered in the 25 secs. will be represented by the area A C D B.

This can be estimated as follows :

$$\text{Area of } \triangle\ ACH = \frac{10 \times 10}{2} = 50 \text{ ft.}$$

$$\text{Area of rectangle } CDJH = 10 \times 10 = 100 \text{ ft.}$$

$$\text{Area of } \triangle\ DJB = \frac{10 \times 5}{2} = 25 \text{ ft.}$$

$$\text{Total distance covered} = \underline{\underline{175}} \text{ ft.}$$

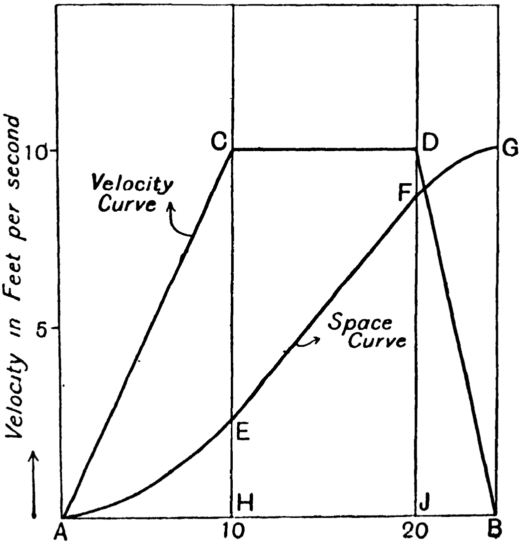


Fig. 16.

The space curve will be as indicated, A E and F G being parabolic arcs and E F a straight line. As an exercise the reader should draw the curve by the sum curve construction.

Suitable scales would be as follows : Time, 1″ = 5 secs. ; velocity, 1″ = 5 ft. per sec. ; polar distance = 2 in.

Then the space scale will be 1″ = 2 × 5 × 5 = 50 ft., so that B G should measure 3·5 in.

**Acceleration.**—When the velocity of a body is changing

it is said to have an *acceleration.* Acceleration is measured by the rate of change of velocity, and it is clear that such change of velocity may take place in magnitude or direction, or both. For the present we will confine ourselves to change of magnitude of velocity and assume that the direction remains constant.

Suppose that a body is moving at a certain instant with a velocity of 10 ft. per sec., and that one second later it is moving in the same direction with a velocity of 12 ft. per sec. In one second the velocity has gained by 2 ft. per sec., so we say that the mean acceleration is 2 ft. per sec. per sec.; this is often written for brevity 2 ft./sec.$^2$

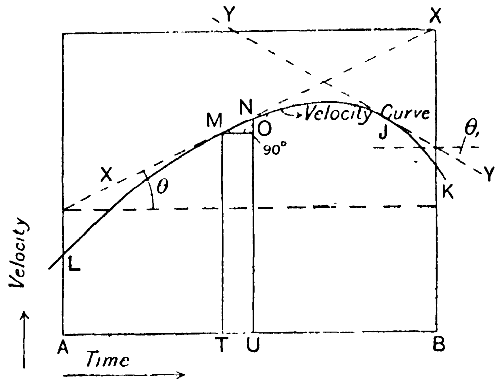


FIG. 17.—Acceleration.

Now let the velocity curve be L M N K, Fig. 17. At the time represented by the point T the velocity is represented by T M, and after a short time T U it is represented by U N, so that in time T U the point has gained in velocity by an amount N O.

$$\therefore \text{ Mean velocity gained in unit time} = \frac{\text{N O}}{\text{M O}}.$$

Now the points T U are very close together, and as N comes closer still to M the line joining M N ultimately becomes the tangent X X to the velocity curve at the point M.

Then rate of change of velocity

$$= \frac{N\,O}{M\,O} = \text{slope of tangent } X\,X = \tan\theta.$$

*Therefore the acceleration at any point is represented by the ope of the velocity curve at the given point.*

If we obtain the accelerations at a number of points and lot them against the times, the resulting curve will be an cceleration curve.

*Positive and Negative Acceleration.*—We have up to the resent spoken only of velocity gained, but the term "gain" ıust be considered as including "loss," and when there is loss we shall regard it as a negative gain. Returning to ır numerical illustration, suppose that instead of being 2 ft. per sec., at the end of one second the velocity is 8 ft. per c.; in one second the velocity has lost 2 ft. per sec., and e should say that the mean acceleration is – 2 ft. per sec. ɔr sec.

Now when the velocity is decreasing, the tangent, such s Y Y, Fig. 17, cuts the base at a point in advance of the ɔint of contact; whereas when the velocity is increasing, ıe tangent cuts the base behind the point of contact. This ıables us to formulate the following rule: If the tangent the velocity curve cuts the time base at a point behind ıe point of contact, the acceleration is positive; and if it ıts at a point beyond the point of contact, the acceleration negative.

**General Relation between Acceleration, Velocity, ıd Space Curves.**—We have shown that the slope at any ɔint of the velocity curve determines the acceleration, ıd we have previously shown that the slope at any point the space curve gives the velocity; there is, therefore, ıe same relation between the acceleration and velocity ırves as there is between the velocity and the space curves. 'e thus get the following very important rule:

*The velocity curve is the sum curve of the acceleration curve, ıd the space curve is the sum curve of the velocity curve.*

This is illustrated in Fig. 18, in which, to save confusion, the three curves have been drawn upon separate bases. C D F is the acceleration curve; drawing a sum curve with polar distance $p_1$, we get the velocity curve H K J; if the point has an initial velocity $v_0$, we set up $A_1H = v_0$ on the

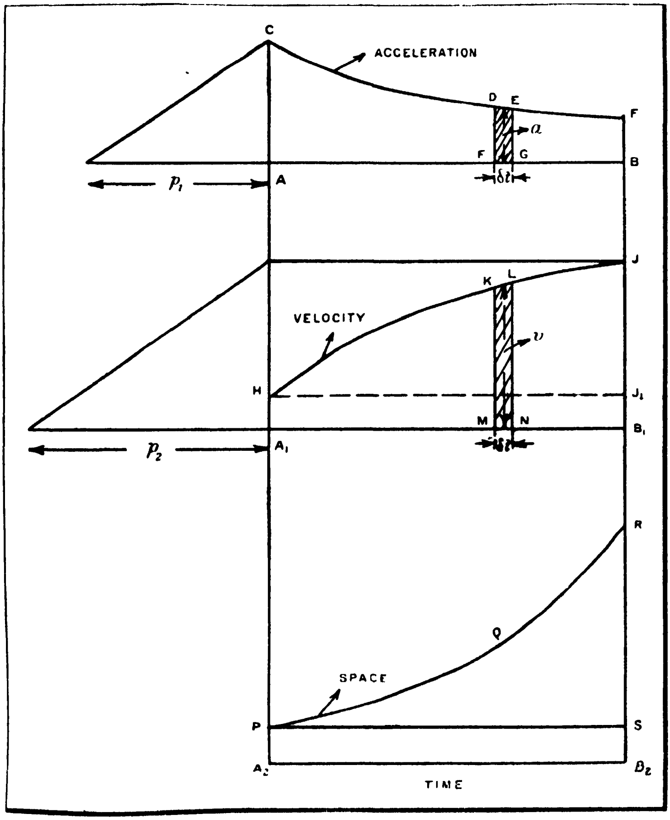


FIG. 18.—Relation of Acceleration, Velocity, and Space Curves.

velocity scale, obtained as described later, and start the sum curve at H. Drawing the sum curve of this with a polar distance $p_2$, we get the space curve P Q R.

SCALES.—Suppose that the time scale is $1'' = x$ secs., and that the acceleration scale is $1'' = y$ ft./sec.$^2$; and suppose

that $p_1$ is measured in actual inches. Then the velocity scale will be $1'' = p_1 x y$ ft./sec. Now let $p_2$ be also measured in actual inches. Then the space scale will be

$$1'' = p_2 p_1 x^2 y \text{ ft.}$$

This may be explained as follows: One square inch of the acceleration curve represents $x y$ units of velocity, and the sum curve construction gives the area divided by the polar distance, so that 1 in. on the sum curve H K J represents $p_1 x y = z$, say.

By similar reasoning, 1 in. on the sum curve P Q R represents $p_2 x z = p_1 p_2 x^2 y$.

As a numerical illustration, let the time scale be $1'' = 10$ secs., and the acceleration scale $1'' = 2$ ft. per sec. per sec.; and let $p_1 = 2$ in.; then the velocity scale will be

$$1'' = 2 \times 2 \times 10 = 40 \text{ ft. per sec.}$$

Next let $p_2 = 1\frac{1}{2}$ in.; then the space scale will be

$$1'' = 1\tfrac{1}{2} \times 40 \times 10 = 600 \text{ ft.}$$

By a careful choice of the polar distances $p_1$, $p_2$ we can obtain convenient scales. We should, for instance, have done better in the above case to have taken $p_1 = 2\frac{1}{2}$ in. and $p_2 = 2$ in.; our velocity scale would then be $1'' = 50$ ft. per sec., and the space scale $1'' = 1{,}000$ ft.

Numerical Example of a Tramcar Starting from Rest.—Fig. 19 shows the result of some starting tests upon a tramcar which was designed to give an initial acceleration of 3 miles per hour per sec. The test was made by noting the times up to varying distances, the results being plotted in the form of the space curve A.

To get from this the velocity diagram we have to find the slope curve of the curve A; there is no simple construction which gives the slope curve accurately. One of the best ways to proceed is to take short time intervals, say one second each, one each side of a number of main ordinates, and to find the corresponding points, such as $a b$, on the curve; by drawing

$bc$ horizontally we get slope of space curve $= \dfrac{bc}{ac} = \dfrac{bc}{2}$ in our case = velocity $de$.

The velocities thus found for a number of points are then plotted to give the velocity curve B. In the present case at 15 secs. $bc = 88$ ft., so that $de = \dfrac{88}{2} = 44$ ft. per sec. = 30 miles per hour.

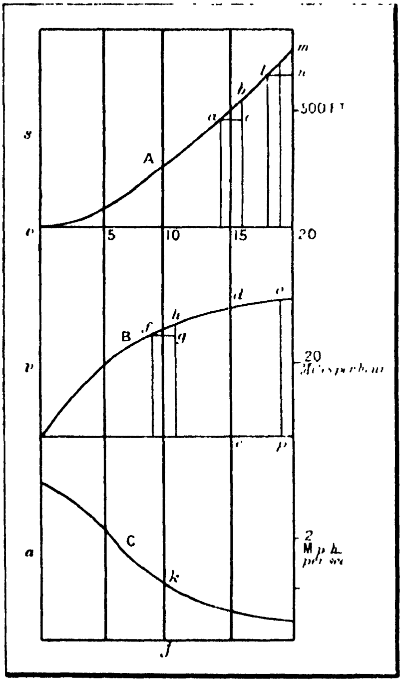


FIG. 19.—Example of Tramcar Starting from Rest.

At the 20-sec. point we cannot take points on either side, so it is better to calculate the velocity at 19 secs. by taking $op = \dfrac{mn}{2}$; the velocity curve is then continued past $o$ to the end.

To draw the acceleration curve C we work in a similar

manner from the velocity curve; at 10 secs., for instance, we have acceleration = slope of velocity curve $= \frac{hg}{fg} = \frac{hg}{2}$, so that by plotting up $jk$ equal to $\frac{hg}{2}$ to an enlarged scale, we get a point $k$ on the acceleration curve, and, by joining up such points, we get the completed curve C.

**Constant Acceleration; Equations of Motion.**—We will now show how to derive equations, for the case in which acceleration is constant, from consideration of the various

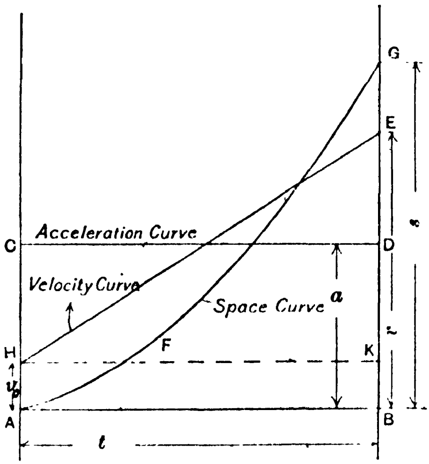


FIG. 20.

curves. One important feature of the study of engineering theory from the graphical standpoint, which many authorities have failed to appreciate, is the elegant manner in which formulæ can be derived to deal with the simplest cases. It does not at all follow that, in dealing with the subject from graphical considerations, we must necessarily employ purely graphical methods in order to make our calculations; we derive formulæ for the simplest cases, and resort to graphical constructions only when formulæ are inadequate or their use does not save time.

If the acceleration is constant, the acceleration curve is a horizontal straight line C D, Fig. 20; the sum curve, *i.e.* the velocity curve of this, will be the sloping straight line H E, while the space curve A F G will be a parabola.

From these curves we can deduce the following formulæ:

$$\text{K E} = \text{area of acceleration curve} = a\,t;$$

$$\therefore\ v = \text{B K} + \text{K E}$$

$$= v_0 + a\,t \quad \dots\dots\dots\dots\dots\dots\dots \quad (5)$$

$$s = \text{area A H E B}$$

$$= t\,\frac{(v_0 + v)}{2} = t\left(\frac{v_0 + v_0 + a\,t}{2}\right)$$

$$= v_0 t + \tfrac{1}{2} a\,t^2 \quad \dots\dots\dots\dots\dots\dots\dots \quad (6)$$

We can get a third relation as follows:

By squaring equation (5) we have

$$v^2 = (v_0{}^2 + a\,t)^2 = v_0{}^2 + 2\,v_0 a\,t + a^2 t^2$$

$$= v_0{}^2 + 2\,a\,(v_0 t + \tfrac{1}{2}\,a\,t^2)$$

$$= [\text{from } (6)]\ v_0{}^2 + 2\,a\,s \quad \dots\dots\dots\dots\dots \quad (7)$$

These equations (5) to (7) are often called the equations of motion and are very useful in problems in which the acceleration is constant.

NUMERICAL EXAMPLES.—(1) *A point moves along a straight line under an acceleration of* 10 *ft./sec.*² *The initial velocity is* 7 *ft./sec. What is the velocity after it has passed over* 12 *ft.?*

In this case $u = 7$ ft. per sec.,

$a = 10$ ft. per sec. per sec.,

$s = 12$ ft.

Therefore, using equation (7),

$$v^2 = 7^2 + 2 \times 10 \times 12$$

$$= 49 + 240$$

$$= 289,$$

$$v = \sqrt{289} = 17 \text{ ft. per sec.}$$

(2) *A train is running at* 20 *miles an hour and is stopped by brakes in* 10 *secs., the retardation being constant. At how many yards from the stopping point were the brakes applied?*

$$60 \text{ miles an hour} = 88 \text{ ft. per sec.}$$

$$\therefore\ 20 \text{ miles an hour} = \frac{88}{3} \text{ ft. per sec.}$$

In this case $v_0 = \frac{88}{3}$ ft./sec., $t = 10$, and $v = 0$;

$$\therefore\ 0 = \frac{88}{3} + 10a \quad [\text{from } (5)];$$

$$\therefore\ a = -\frac{88}{30} \text{ ft./sec.}^2;$$

$$\begin{aligned}\therefore\ s &= v_0 t + \tfrac{1}{2} a t^2 \\ &= \frac{88}{3} \times 10 - \frac{88}{2 \,.\, 30} \,.\, 100 \\ &= \frac{880}{3}(1 - \tfrac{1}{2}) = \frac{440}{3} \text{ ft.} = \frac{440}{9} \text{ yds.} \\ &= \underline{48{\cdot}89 \text{ yds.}}\end{aligned}$$

**Gravity Acceleration "g."**—If bodies are allowed to drop freely they will be found to have an acceleration which is practically constant.

This acceleration is called the gravity acceleration and given the letter $g$. Its value varies slightly with the latitude and with the height above sea-level, and in London is usually taken as 32·2 ft. per sec. per sec. We will now derive simplified formulæ for the case of bodies falling freely from rest. In equations (5) to (7), therefore, we have $v_0 = 0$ and $a = g$, and it is usual to replace the distance or space $s$ by the height $h$. Our formulæ, therefore, become

$$v = g\,t \quad \dots \quad (8)$$

$$h = \tfrac{1}{2}\,g\,t^2 \quad \dots \quad (9)$$

$$v^2 = 2g\,h \quad \dots \quad (10)$$

Formula (10) is of the greatest possible importance and may be rewritten in the forms

$$v = \sqrt{2g\,h} \quad \dots \quad (11)$$

$$h = \frac{v^2}{2g} \quad \dots \quad (12)$$

**Measurement of Kinetic Energy.**—We have already indicated that kinetic energy is the amount of work stored in a body in virtue of its velocity, but we have not yet explained how this energy can be measured.

Suppose that a body of weight W starts from rest and falls a vertical distance $h$ without overcoming any resistance; then the weight, which is the force acting on the body, has done an amount of work equal to force $\times$ distance moved in the direction of the force $=$ W $h$. Since no work has been done in overcoming resistance, the whole of this work must be stored up in the body in the form of kinetic energy (K.E.); and since the body was originally at rest and possessed no kinetic energy, it follows that the latter must after falling be equal to W $h$, *i.e.*

$$\text{K.E.} = \text{W}\,h \quad \ldots\ldots\ldots\ldots \quad (13)$$

But we have already shown that for bodies falling freely under the action of gravity

$$v^2 = 2\,g\,h,$$

*i.e.*
$$h = \frac{v^2}{2\,g};$$

$\therefore$ we have
$$\text{K.E.} = \frac{\text{W}\,v^2}{2\,g} \quad \ldots\ldots\ldots\ldots \quad (14)$$

This is a very important formula. In using it, we must note that it does not matter how the body has moved in obtaining this velocity; all that matters is that the body, somehow or other, has attained a velocity $v$. Then we say that its K.E. is $\dfrac{\text{W}\,v^2}{2\,g}$.

It is a fundamental principle of mechanics that the work done by a force depends only on the straight distance in the direction of the force between the original and final positions of the body. If, for instance, the body had moved in an irregular path, the work stored in it would still have been W $h$, and, therefore, the kinetic energy would still be W $h$; and since the kinetic energy depends only on the velocity

by definition, it must be equal to $\frac{W v^2}{2g}$, whatever be the path traversed.

The direction of the velocity will be different in the two cases, but that does not matter so far as kinetic energy is concerned.

*Change in Kinetic Energy.*—Suppose that a body of weight W has at one instant a velocity $v_0$, and at some subsequent instant a velocity $v$.

Then its kinetic energy has changed from $\frac{W v_0^2}{2g}$ to $\frac{W v^2}{2g}$.

$$\therefore \text{Change in K.E.} = \frac{W}{2g}(v^2 - v_0^2) \quad \ldots\ldots\ldots (15)$$

**The Connection between Force and Acceleration.**—Suppose that a body of weight W is moved by an effort E a very short distance $s$ in the direction of E, and that its velocity at the beginning of the distance is $v_0$, and at the end $v$.

Then work done = E . $s$.

If this all goes in increasing the K.E. we have

$$E\,.\,s = \text{gain in K.E.}$$
$$= \frac{W}{2g}(v^2 - v_0^2) \quad \ldots\ldots\ldots\ldots\ldots (16)$$

Now if $s$ is so short that the force E is constant over it and that the acceleration is also constant, we have, by formula (7), p. 32,

$$v^2 = v_0^2 + 2as,$$

*i.e.*
$$v^2 - v_0^2 = 2as.$$

Putting this in (16),

$$E\,.\,s = \frac{W}{2g}\,.\,2as,$$

*i.e.*
$$\therefore E = \frac{Wa}{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

The above formulæ, which we have deduced from considering the space, velocity, and acceleration curves, are of funda-

mental importance, and will be used frequently in later portions of the book.

NUMERICAL EXAMPLE.—*The piston of an engine weighing 20 lb. is given a retardation of 6 ft. per sec. per sec. What backward pressure will be acting on the piston?*

We shall discuss the crank and connecting-rod mechanism of an ordinary engine in a later chapter (p. 115), and those students who would like to understand this question with special reference to its application are recommended to refer to that description.

Putting our values in equation (17),

$$E = -\frac{Wa}{g} = -\frac{20 \times 6}{32 \cdot 2} \quad \text{(}-\text{ indicates backward pressure)}$$

$$= \underline{3 \cdot 73 \text{ lb.} \quad \textit{Ans.}}$$

## CHAPTER III

### CURVES OF ACTION AND CONSTRUCTIONS THEREFOR

We have considered curves of velocity plotted against time; we will now consider curves in which velocity is plotted against space or distance, such curves being referred to as *curves of action.*

Going back to the consideration of effort and work curves, we see that if the effort diagram represents resultant effort (*i.e.* effort *minus* resistance), the whole of the work done upon the body must go in increasing its velocity, *i.e.* in increasing kinetic energy; and we can calculate from the work done up to any point the velocity which the body should have at that point.

If $v_0$ ft. per sec. is the velocity of the body at the starting point O, and $v$ that at any other point at which the ordinate of the work diagram is $e$, we have, if W lb is the weight of the body, $e$ = increase of kinetic energy

$$= \frac{Wv^2}{2g} - \frac{W v_0^2}{2g}, \qquad \text{(1)}$$

$g$ being the gravitational acceleration, which is 32·2 ft. per sec. per sec.

$$\therefore \ v^2 = v_0^2 + \frac{2ge}{W} \qquad \text{(2)}$$

From this equation the velocity $v$ can be calculated at any point, and from these results we could plot a fresh diagram of velocities upon a distance base, thus obtaining the curve of action.

**Auxiliary Parabola Construction for Curve of Action from Curve of Work.**—Let O B C, Fig. 21, be a curve of work, and let the ordinate B D at any point D be $e$. If $v_0$ is the velocity at the point O, set down on the work scale a length O E to represent $\frac{W v_0^2}{2g}$, and set out O F on A O produced to represent the initial velocity $v_0$ to a con-

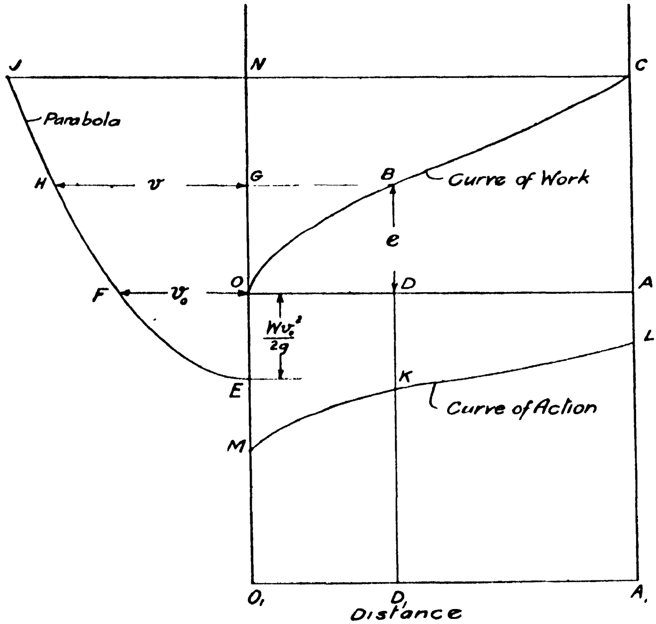


FIG. 21.—Auxiliary Parabola Construction.

venient scale. Then with vertex E and axis O E draw a parabola E F J to pass through the point F and extending to the horizontal line C J passing through the highest point of the curve of work.

Then from the property of the parabola we have that, if G H is the abscissa of the parabola corresponding to the point B on the curve of work,

$$\frac{GH^2}{OF^2} = \frac{EG}{EO} = \frac{EO + OG}{EO} = 1 + \frac{OG}{EO}.$$

*i.e.*
$$GH^2 = v_0^2\left(1 + \frac{OG}{EO}\right)$$
$$= v_0^2 + \frac{v_0^2 \cdot e}{\frac{W v_0^2}{2g}}$$
$$= v_0^2 + \frac{2g \cdot e}{W}.$$

We see, therefore, from equation (2) that $GH = v$. If, therefore, we make $O_1 M = OF$ and $D_1 K = GH$, and so on for a number of points we obtain the curve of action M K L.

**Relation between the Curve of Resultant Effort and the Curve of Action.**—We have just seen how the velocity at any point can be calculated from the curve of work; we will now consider how the curve of resultant effort may be obtained direct from the curve of action.

We have already reminded ourselves of the fundamental law of dynamics that

$$\text{Effort or force} = \frac{\text{weight} \times \text{acceleration}}{\text{gravity acceleration}}, \ i.e.\ E = \frac{Wa}{g}.$$

The curve of resultant effort, therefore, can always be converted into a curve of acceleration by changing the scale in the ratio $\frac{g}{W}$.

If, for instance, the curve of resultant effort is to scale $1'' = 1{,}000$ lb. (1 kip), and the weight of the body is 2,000 lb. (2 kips), the acceleration scale would be

$$1'' = \frac{1{,}000 \times 32 \cdot 2}{2{,}000} = 16 \cdot 1 \text{ ft./sec.}^2$$

Now let P, Q, Fig. 22, be two points very near to each other on the curve of action, so near, in fact, that the straight line P Q may be taken in place of the curved arc, and the line R N, which is perpendicular to the mid-point R of P Q, may be taken as the normal to the curve.

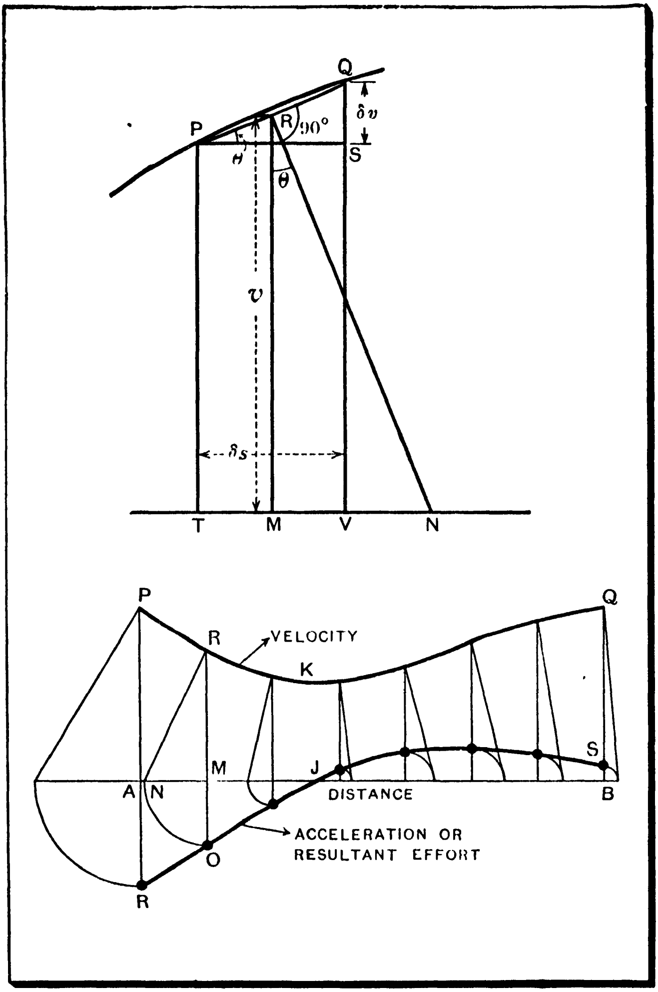


FIG. 22.—Curve of Resultant Effort from Curve of Action.

We then note that

$$\tan\theta = \frac{\mathrm{QS}}{\mathrm{PS}} = \frac{\delta v}{\delta s}$$

$$\text{also } \tan\theta = \frac{\mathrm{MN}}{\mathrm{MR}} = \frac{\mathrm{MN}}{v};$$

$$\therefore\ \mathrm{MN} = \frac{v\,.\,\delta v}{\delta s} \quad\ldots\ldots\ldots\ldots\ldots \quad (3)$$

But we have shown that the formula for motion under constant acceleration is

$$v^2 = v_0{}^2 + 2\,a\,s,$$

where $v$ = final velocity and $v_0$ = initial velocity.

Applying this formula to Fig. 22, which is justifiable since the length P Q is so short that the acceleration may be taken constant over it, we have

$$\mathrm{QV}^2 = \mathrm{RM}^2 + 2\,a\,\mathrm{MV},$$

*i.e.*

$$\left(v + \frac{\delta v}{2}\right)^2 = v^2 + 2\,a\,.\,\frac{\delta s}{2}$$

$$v^2 + v\,.\,\delta v = v^2 + a\,.\,\delta s \ \left(\text{neglecting } \frac{\delta v^2}{4}\right),$$

*i.e.*

$$a = \frac{v\,.\,\delta v}{\delta s}; \quad\ldots\ldots\ldots\ldots\ldots \quad (4)$$

$\therefore$ [from equation (3)] $\mathrm{MN} = a$.

But MN is the subnormal to the curve of action at the point R, so that we get the following rule: *The acceleration of a body at any point of its motion is given by the subnormal to the curve of action.* Where the acceleration is found we have only to multiply by the quantity $\frac{W}{g}$ to get the resultant effort.

To get, therefore, the curve of acceleration from the curve of action we proceed as follows:

Let P R Q represent the curve of action, *i.e.* a curve of velocities plotted upon a distance base.

At convenient distances apart take a number of points, such as R, upon the curve, and at each point draw lines, such as R N, at right angles to the tangent to the curve at the point and with centre M; describe an arc M N to meet R M

(produced if necessary) in O. Then M O is the acceleration at the point. When N comes to the left of M the acceleration is negative and the arc is swung downwards, but when it comes to the right it is positive and is swung upwards.

In this way, by joining up points such as O, we get the curve R O S of acceleration, which, as explained above, becomes the curve of resultant effort by a suitable change of scale. This is sometimes referred to as "Pröll's construction."

It will be noted that between R and J the acceleration is negative and, therefore, what is commonly called a retardation. This means that the resultant effort is negative, so that the gross effort is less than the resistance. From J to S the acceleration is positive, so that here the gross effort is greater than the resistance. If, for instance, the curves shown relate to a tramcar or railway train, between R and J the brakes are acting, but are being gradually taken off so that the velocity continues to diminish up to the point K; from J onwards the driving force or gross effort is greater than the resistance to the motion, so that the velocity increases. Towards S it will be noticed that the acceleration, and, therefore, the resultant effort, begin to fall off somewhat. If running on the level, this is probably due to the fact that air resistance increases rapidly with the increase of velocity, and as the body speeds up the resistance increases, although the full driving effort may continue to be exerted.

SCALES.—In the above construction let the space scale be $1'' = y$ ft., and the velocity scale be $1'' = z$ ft. per sec.; then the acceleration scale will be $1'' = \dfrac{z^2}{y}$ ft. per sec. per sec., so that if the space scale is $1'' = 10$ ft., and the velocity scale $1'' = 20$ ft. per sec., the acceleration scale will be

$$1'' = \frac{20 \times 20}{10} = 40 \text{ ft. per sec. per sec.}$$

**Construction for Curve of Action from the Curve of Resultant Effort.**—In our last section we showed that

by drawing the subnormal to the curve of action we can obtain the acceleration, and thus the resultant effort at any point, so that we can get the resultant effort curve from the curve of action by a very simple construction. We will now give the construction for the inverse operation, which is not quite so simple, but which does not present any particular difficulty.

Before we can start this construction we must know the velocity $v_0$ which the body has at the commencement of the period under consideration. This need for knowing the initial value is analogous to the familiar fact that in the construction for bending moment diagrams in graphic statics we must know how the ends of the beams are supported before we can complete the construction. We will describe two constructions which are similar in principle, but differ slightly in application.

(1) *Circular Arc Method.* Let $a'$, $b'$, $c'$, etc., Fig. 23, represent the curve of resultant effort corresponding to which the curve of action is required. Divide up the base into a convenient number of small portions, preferably equal, and draw the mid-ordinates $a\,a'$, $b\,b'$, etc., of each.

At some convenient place, preferably vertically below the curve of resultant effort, take a base O X, and mark out the mid-points $a$, $b$, $c$, etc.; then from each of these mid-points set out lengths $a\,a'$, $b\,b'$, $c\,c'$, etc., equal to the ordinates $a\,a'$, etc., of the curve of resultant effort. Beyond the point $d$ the resultant effort curves become negative, so that the distances are then set out in the opposite direction.

Now set up O $h$ to represent the initial velocity $v_0$, and with centre $a'$ strike an arc $h\,j$ across the first space $a$; next take $b'$ as centre, and strike an arc $j\,k$ across the space $b$, and so on.

The resulting curve is a series of circular arcs (which can be smoothed out, if necessary, with the pear curve) which gives the curve of action.

The proof is very simple, because the radius of a circular

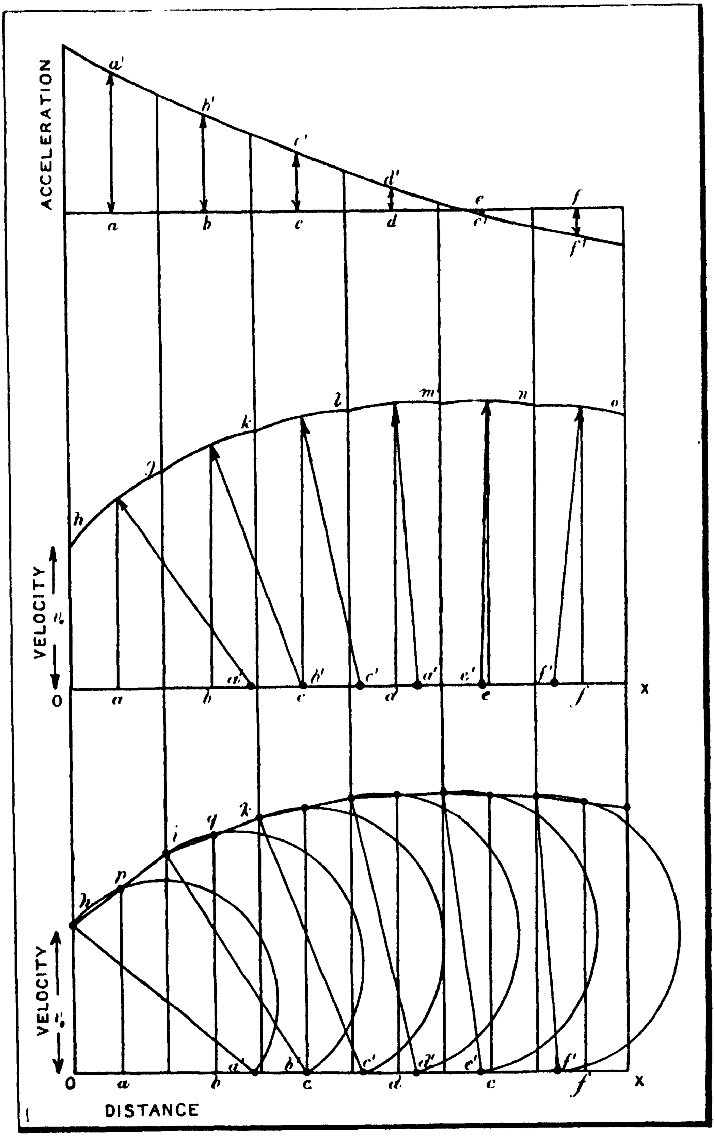


Fig. 23.—Circular Arc and Semicircle Methods of Construction of Curve of Action from Curve of Resultant Effort.

arc is the normal to the curve, so that $a\,a'$, etc., are subnormals to this curve, and as these were made equal to the resultant efforts, the necessary relation between the two curves is maintained. For accuracy, the divisions should be taken as small as possible, particularly where the curve of action tends to cut the base. At such a point the velocity becomes negative, or, in other words, the body begins to run backwards.

(2) *Semicircle Method.*—This is shown on the lower portion of Fig. 23. Set out the points $a'$, $b'$, etc., as before and also the point $h$. Then join $a'\,h$ and draw upon it a semicircle, cutting the first mid-ordinate in $p$, and join $h\,p$ and produce to meet the second main ordinate in $i$. Now join $i\,b'$ and on it describe a semicircle cutting the second mid-ordinate in $q$ and join $i\,q$, producing it to meet the third main ordinate in $k$, and so on. Then the figure $h$, $i$, $k$, etc., is the curve of action required, the accuracy of the construction becoming greater as the number of base elements or divisions is increased.

The proof of this construction is based upon the fact that the angle in a semicircle is a right angle, so that $p\,a'$ is perpendicular to $h\,i$, and so is normal to the curve, so that as $a\,a'$ was made equal to the resultant effort, the necessary relation between the two curves is maintained.

Scales.—The determination of the scales in this case is similar to that in the converse construction. Let the curve of the resultant effort be drawn to a scale 1 in. $= u$ lb., and let W lb. be the weight of the moving body; the acceleration scale is then 1 in. $= \dfrac{u\,.\,g}{\mathrm{W}}$ ft. per sec. per sec. Let the space scale be as before, 1 in. $= z$ ft.; then the velocity scale will be 1 in. $= \sqrt{\dfrac{z\,.\,u\,.\,g}{\mathrm{W}}}$ ft. per sec.

This may be expressed in words as follows: ***The velocity scale is the geometric mean between the space scale and the acceleration scale.***

To make this quite clear we will consider the following case: Suppose that the curve of resultant effort is drawn to a scale 1 in. = 100 lb. and the body weighs 500 lb.; and let the space scale be 1 in. = 10 ft.; then, taking $g = 32$ ft. per sec., we get

$$\text{Velocity scale} = 1 \text{ in.} = \sqrt{\frac{100 \times 32 \times 10}{500}}$$

$$= \sqrt{6\cdot4} = 2\cdot5 \text{ ft. per sec.}$$

**Special Cases of Curves of Action.**—(1) *Resultant Effort or Acceleration Constant.*—In this case the subnormal to the curve of action is of constant length, and it is one of the properties of the parabola that its subnormal is of constant length, so that in this case the curve of action is a parabola. To get the vertex O to enable us to draw the parabola in the case in which the body does not start from rest we proceed as follows:

Set up A C, Fig. 24, equal to the initial velocity $v_0$, and make A D equal to the constant acceleration $a$. Join D C and draw C E at right angles to C D, cutting the base produced in E, and bisect A E in O, which gives the vertex required. The proof of this follows from the property of the parabola, that the tangent of a parabola at any point intersects the axis at a distance beyond the vertex equal to the ordinate of the point measured from the vertex, *i.e.* O E = O A.

Let F A = $v$ be the ordinate at the point F, the corresponding abscissa or space of which is $s$.

We then have from the property of the parabola

$$\frac{\text{F H}^2}{\text{A C}^2} = \frac{\text{O F}}{\text{O A}},$$

*i.e.*
$$\frac{v^2}{v_0^2} = \frac{\text{O A} + \text{A F}}{\text{O A}} = 1 + \frac{\text{A F}}{\text{O A}} \quad \dots\dots\dots\dots (5)$$

Now, since D C E is a right angle, we have

$$\text{A C}^2 = \text{A E . A D},$$

*i.e.*
$$v_0^2 = 2 \times \text{O A} \,.\, a,$$

or $$\mathrm{OA} = \frac{v_0^2}{2a};$$

$$\therefore \frac{\mathrm{AF}}{\mathrm{OA}} = s \div \frac{v_0^2}{2a} = \frac{2as}{v_0^2};$$

$$\therefore \text{[from (5)]}\ v^2 = v_0^2\left(1 + \frac{2as}{v_0^2}\right)$$

$$= v_0^2 + 2as \quad \ldots\ldots\ldots\ldots \quad (6)$$

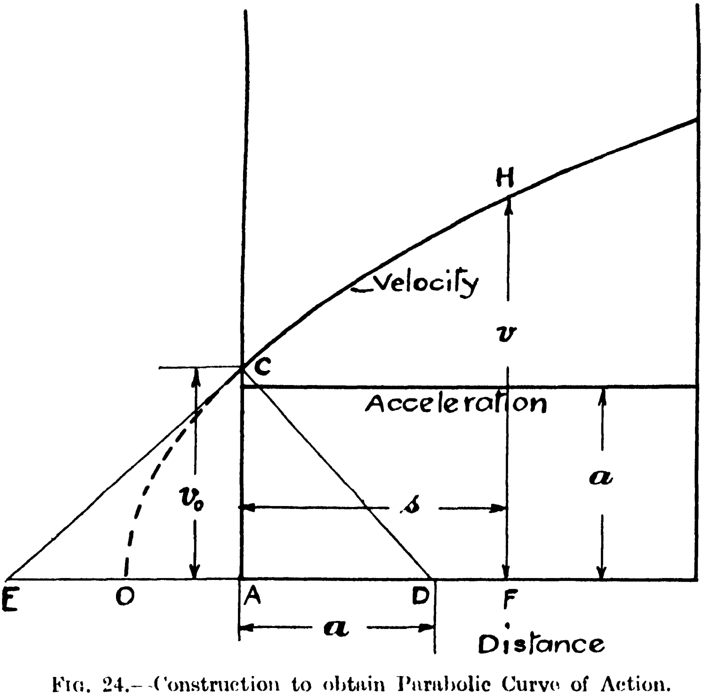


FIG. 24.—Construction to obtain Parabolic Curve of Action.

This is the same formula as that which we have already obtained on p. 32 from other considerations.

(2) *Simple Harmonic Motion.*—Simple harmonic motion may be defined as "the motion of a body such that the force or resultant effort acting upon the body in the direction of motion is proportional to the distance of the body from a certain point called the centre of motion and acts in a direction opposite to the direction of motion of the body."

Referring to Fig. 25, therefore, the curve of resultant effort for simple harmonic motion will be the straight line C D, O being the centre of motion.

Bearing in mind the relation between the two curves, we see that the subnormal to the curve of action at any point G has to be proportional to the ordinate E of the corresponding point F of the curve of resultant effort, and therefore proportional to the abscissa S.

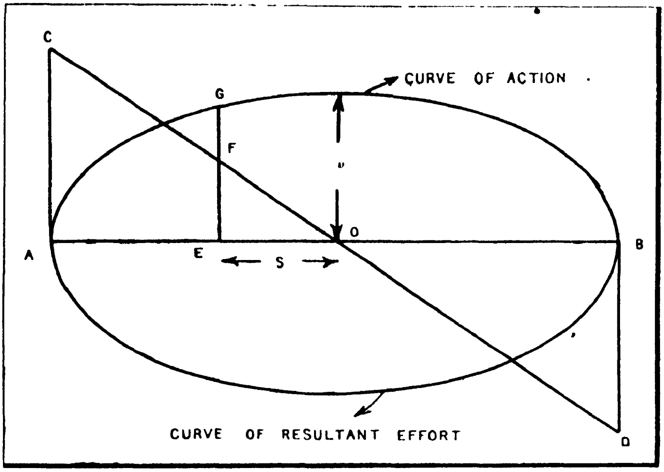


Fig. 25.—Curve of Resultant Effort for Simple Harmonic Motion.

A curve whose subnormal is proportional to the abscissa of the point is the ellipse, so that *for simple harmonic motion the curve of action is an ellipse.*

The maximum velocity $v$ can be calculated as follows:

Work done in moving from A to O

$$= \text{area of } \triangle\, ACO = \tfrac{1}{2} AC \,.\, AO.$$

All this appears as kinetic energy, so that if W is the weight of the body,

$$\frac{Wv^2}{2g} = \tfrac{1}{2} AC \,.\, AO;$$

$$\therefore\ v = \sqrt{\frac{AC \,.\, AO \,.\, g}{W}}.$$

Or, if $AC = m\,AO$, where $m$ is a constant,

$$v = AO\sqrt{\frac{m g}{W}}.$$

The constant $m$ is the force at unit displacement.

We shall return to this problem later. We will point out at this stage that where a graphical construction involves an ellipse we need not draw the ellipse, which is awkward to draw accurately, but may instead draw a circle of diameter A B in this case, and read the ordinates to a reduced scale, *i.e.* to a scale of reduction $\frac{r}{OB}$.

NUMERICAL EXAMPLE.—*A body weighs one ton and is lifted vertically by a rope, a spring balance being provided to measure the pull in the rope. There is a constant frictional resistance of 1,000 lb. to the motion of the body, and the values of the pull in the rope, after various distances from the initial position, are as given in the following table :*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Distance in feet | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 |
| Pull in rope ... | 5,580 | 5,450 | 5,260 | 5,020 | 4,810 | 4,600 | 4,430 | 4,270 |

*Find the velocity of the body at every point.*

Fig. 26 shows the effort curve C D E, the scales to which it was drawn prior to reproduction being: Distance, 1″ = 10 ft.; effort, 1″ = 800 lb.

The resistance curve F G is drawn to the same scale, and the difference is treated as the resultant effort curve, the circular arc method giving the curve of action A H K.

By the formula previously given we have velocity scale

$$1'' = \sqrt{\frac{z u g}{W}} = \sqrt{\frac{10 \times 800 \times 32{\cdot}2}{2{,}240}} = 10{\cdot}7 \text{ ft. per sec.}$$

By measurement we find that the velocity at the end of the motion is 57·7 ft. per sec.

**To Draw the Space Curve upon a Time Base from the Curve of Action; the Inverse Sum Curve.**—We now come to a construction known as the *inverse sum curve,* which bears some resemblance to the sum curve construction, and is useful in certain problems. The procedure is exactly

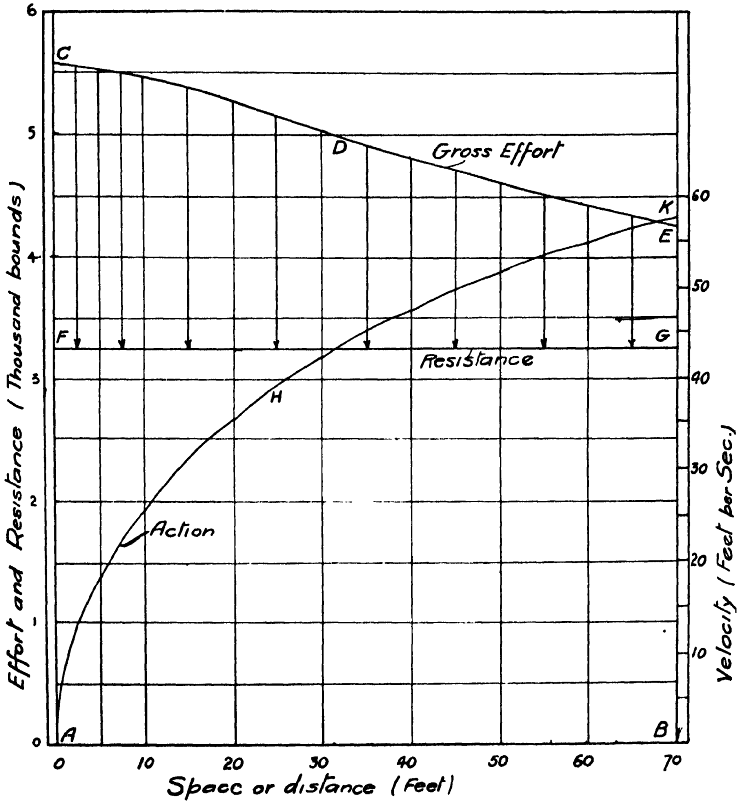


FIG. 26.

the same as for the sum curve up to the finding of the points $1b$, $2b$, $3b$, etc., Fig. 27; we then take a pole X on the opposite side of A to that taken in the sum curve construction, and join up to the points $1b$, $2b$, $3b$, etc.

Across space 1 we then draw A $c$ *at right angles to* X $1b$,

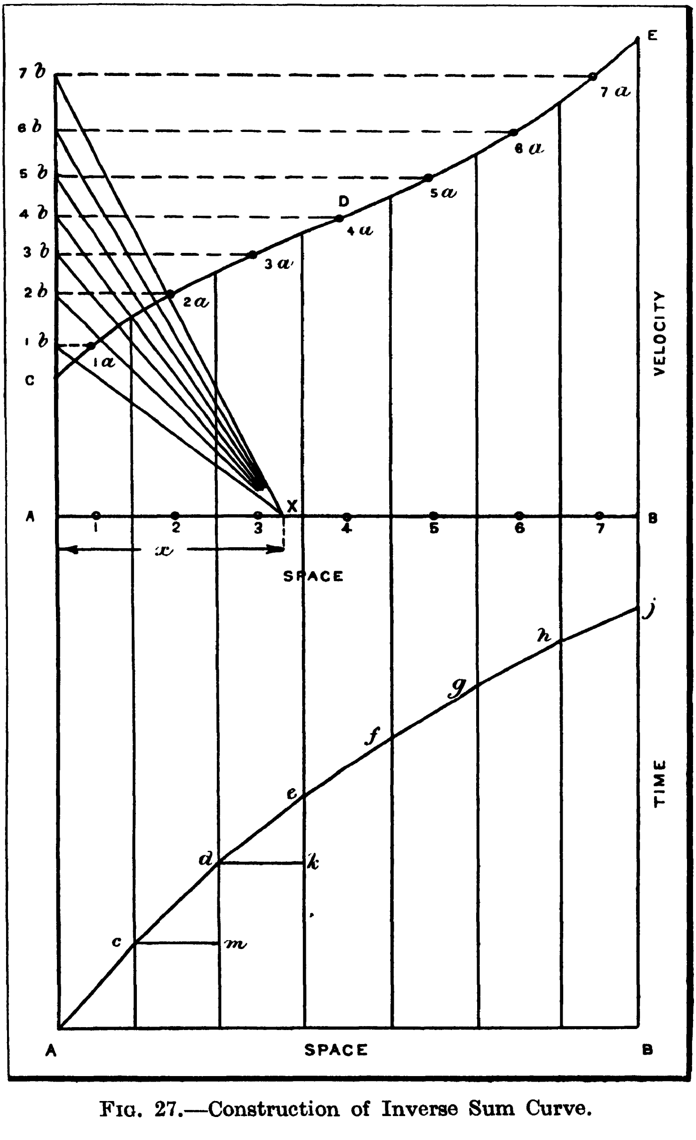


FIG. 27.—Construction of Inverse Sum Curve.

then across space 2, $c\,d$ at right angles to X $2b$, and so on; the resulting curve A $c - j$ is then called the *inverse sum curve.*

Take any one of the spaces, say 3, and draw $d\,k$ horizontal. Then the triangles $d\,e\,k$ and X A $3b$ are similar, because corresponding sides are at right angles to each other.

$$\therefore \frac{e\,k}{d\,k} = \frac{\mathrm{A\,X}}{\mathrm{A}\,3b};$$

$$\therefore e\,k = \frac{d\,k\,.\,\mathrm{A\,X}}{\mathrm{A}\,3b}$$

$$= \frac{d\,k\,.\,x}{3\,.\,3a}$$

$$= \frac{x \times \text{element of space}}{\text{mean velocity over element}};$$

but element of space = element of time $\times$ mean velocity.

$\therefore$ Dividing through by mean velocity, we get

$$e\,k = x \times \text{element of time over 3rd element.}$$

Similarly, $d\,m = x \times$ element of time over 2nd element,

and so on. But ordinate of inverse sum curve at $e$ equals $e\,k + d\,m +$ etc., *i.e.* $x \times$ total time over first three elements. The ordinates, therefore, of the inverse sum curve represent times.

SCALES.—Let the space scale be 1 in. $= y$ ft., and let the velocity scale be 1 in. $= z$ ft. per sec., and let the polar distance be $x$ actual inches; then the time scale will be 1 in. $= \dfrac{y}{x\,.\,z}$ secs.

If, therefore, the space scale is 1 in. = 100 ft., and the velocity scale is 1 in. = 20 ft. per sec., and $x = 2$ in., the time scale will be 1 in. $= \dfrac{100}{40} = 2{\cdot}5$ secs.

**Application to Simple Harmonic Motion.**—Now consider the application of the inverse sum curve construction to the determination of the space curve upon a time base

in the case of simple harmonic motion, for which we have shown, Fig. 25, that the curve of action is an ellipse.

If the inverse sum curve be drawn for this ellipse it will be found to be a curve A E F K, Fig. 28, which is a curve of sines, and as the body moves backwards and forwards

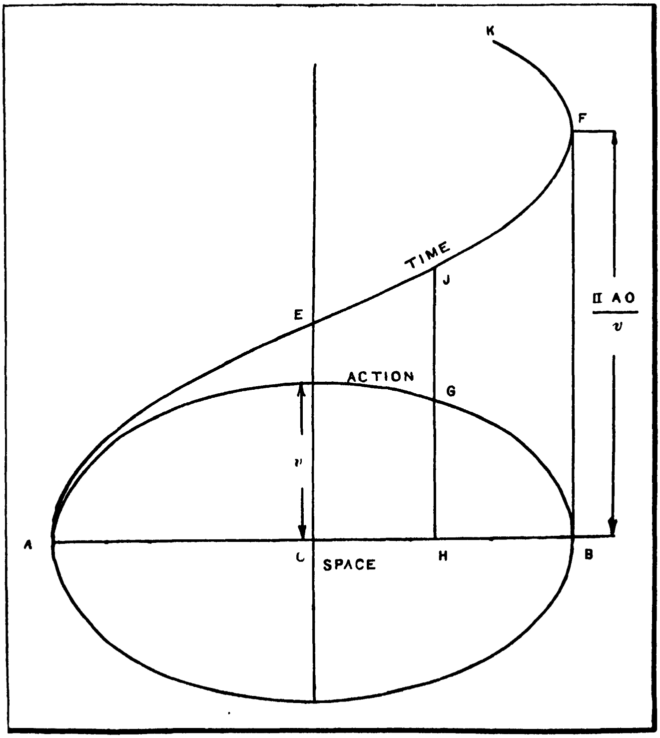


FIG. 28.—Inverse Sum Curve in case of Simple Harmonic Motion.

along A B the curve continues indefinitely by repeating the undulations.

If drawn very carefully, the time B F will be found to be $\frac{\pi \times AO}{v}$; this can be proved mathematically, and we have shown that $v = AO\sqrt{\frac{mg}{W}}$.

Calling this time $t$, we have

$$t = \frac{\pi \times \mathrm{A\,O}}{\mathrm{A\,O}\sqrt{\frac{mg}{\mathrm{W}}}} = \pi\sqrt{\frac{\mathrm{W}}{mg}}.$$

Now the time $2t$, which is the time for a complete movement from A to B and back again, is called the *time period.*

$$\therefore \text{ time period} = 2\pi\sqrt{\frac{\mathrm{W}}{mg}}$$

$$= 2\pi\sqrt{\frac{\text{weight of body}}{g \times \text{force at unit displacement}}}.$$

This formula is very useful for calculating the time of vibration of springs and pendulums.

# CHAPTER IV

## POLAR DIAGRAMS

THE relations between space, velocity, and acceleration diagrams plotted in rectangular or Cartesian co-ordinates have already been considered. In the present chapter we shall deal with polar diagrams for those quantities in which one quantity is measured by the angle turned through by a radius vector from a base or reference line, and the other quantity is measured by the distance along such radius vector from the origin. We shall see that some particular applications of these diagrams result in well-known constructions which are usually studied in an isolated manner without reference to general principles.

**Polar Space, Velocity, and Acceleration Diagrams with Reference to Time.**—We shall deal first with polar diagrams in which angular distances represent time, and the distances along the radius vector represent space, velocity, or acceleration.

Fig. 29 represents a polar space diagram with reference to time. In this diagram O B represents the space $s_0$ of the body under consideration from some starting point at the point from which the times are counted; the angle B O C represents a time $t_1$, by which the space has become $s_1$, the length O C being made equal to $s_1$ to some convenient scale. Similarly, after a time $t_2$, the space $s_2$ is plotted to give the point D. By obtaining a large number of points such as C and D and drawing a smooth curve through such points, we then obtain the polar space curve.

**Construction to Give Velocity at any Point of the Polar Space Curve.**—We can obtain a construction for the

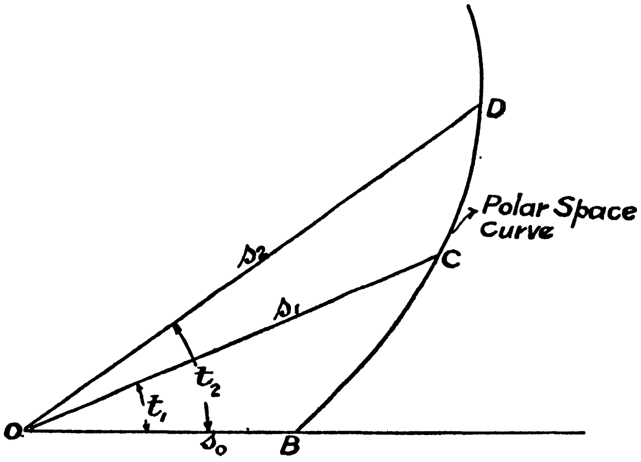


FIG. 29.—Polar Space Curve with reference to Time.

velocity corresponding to any point of the polar space curve by means of the following consideration :

Consider two points B, C, Fig. 30, very near together on the polar space curve, and draw B D at right angles to

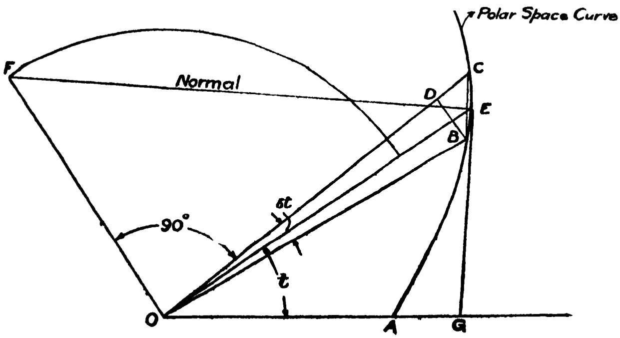


FIG. 30.—Deducing Velocity from Polar Space Curve.

O C. Then if the angle B O C = $\delta t$ is very small indeed, we may take O B = O D, so that if O B = $s$ and O C = $s + \delta s$,

D C will be equal to $\delta s$, which is the increase in space during the short time interval $\delta t$.

Now take the mid-point E of B C and draw E F at right angles to B C, and draw O F at right angles to O E, thus determining the point F. Now as B and C approach closer and closer to each other the line B C approaches more nearly to the tangent E G to the curve, and E F approaches the normal to the curve. Also the $\triangle$s O F E and B C D become similar because their sides are mutually at right angles to each other. In the limit, therefore, we have

$$\frac{\mathrm{O\,F}}{\mathrm{O\,E}} = \frac{\mathrm{D\,C}}{\mathrm{B\,D}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now velocity is defined as the rate of change of space, so that the mean velocity between B and C is given by

$$v = \frac{\text{increase of space}}{\text{increase of time}} = \frac{\delta s}{\delta t}$$

$$= \frac{\mathrm{D\,C}}{\delta t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

And since B D may be replaced by an arc with centre O when the angle $\delta t$ is very small, we have

$$\text{arc B D} = \text{radius} \times \text{angle} = \mathrm{O\,B} \times \delta t.$$

$$\therefore\ \mathrm{B\,D} = \mathrm{O\,B} \times \delta t,$$

*i.e.*
$$\delta t = \frac{\mathrm{B\,D}}{\mathrm{O\,B}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

or
$$v = \mathrm{D\,C} \div \frac{\mathrm{B\,D}}{\mathrm{O\,B}}$$

$$= \frac{\mathrm{D\,C} \times \mathrm{O\,B}}{\mathrm{B\,D}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Therefore, using (1), we see that

$$v = \frac{\mathrm{O\,F}}{\mathrm{O\,E}} \times \mathrm{O\,B}, \quad \ldots\ldots\ldots\ldots\ldots (5)$$

and as the points B C become infinitely close we may put $\mathrm{O\,E} = \mathrm{O\,B}$, thus getting

$$v = \mathrm{O\,F} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

If we swing down O F with centre O on to O E, the resulting intersection with O E is a point on the polar velocity curve. We get, therefore, the following construction for drawing the polar velocity curve when the polar space curve is given:

Take a number of points $S_0$, $S_1$, $S_2$, etc., Fig. 31, upon the polar space curve and draw normals $S_0 F_0$, $S_1 F_1$, etc., to the curve at these points. We have, for convenience of illustration, taken our time base line O A vertically instead of horizontally, as is more usual. This, of course, makes no difference to the construction. Draw $O F_0$ at right angles to $O S_0$ to intersect $S_0 F_0$ in $F_0$, and with centre O swing down $O F_0$ on to $O S_0$ to give the point $V_0$. Then draw $O F_1$ at right angles to $O S_1$ to intersect $S_1 F_1$ in $F_1$, and swing down on to $O S_1$ to give the point $V_1$; and so on for each of the points.

If the points $V_0$, $V_1$, etc., are then joined up by a smooth curve, we get the polar velocity curve for the given polar space curve.

If, instead of swinging the points $V_1$, $V_2$, etc., down on to the corresponding radius vectors, we had joined up the points $F_0$, $F_1$, $F_2$, etc., we should have obtained a curve, shown in dotted lines, which is exactly the same in shape as the curve $V_0$, $V_1$, $V_2$, etc., but is twisted through a right angle. In most cases this is just as convenient in use, because we have only to measure our times from O A′ instead of O A, and this is much facilitated by the method of setting out the times which is shown on the drawings and which we will now describe.

With centre O draw a semicircle (or a whole circle where it is more convenient) and choose a suitable time scale so that the semicircle represents a convenient whole number of units. By means of dividers or by protractor the circumference is then divided up into convenient small equal time intervals A, 1; 1, 2; etc. By repeating these, starting from A′ and making A A′, 1′ = A A, 1′; 1′2′ = 1, 2, etc.,

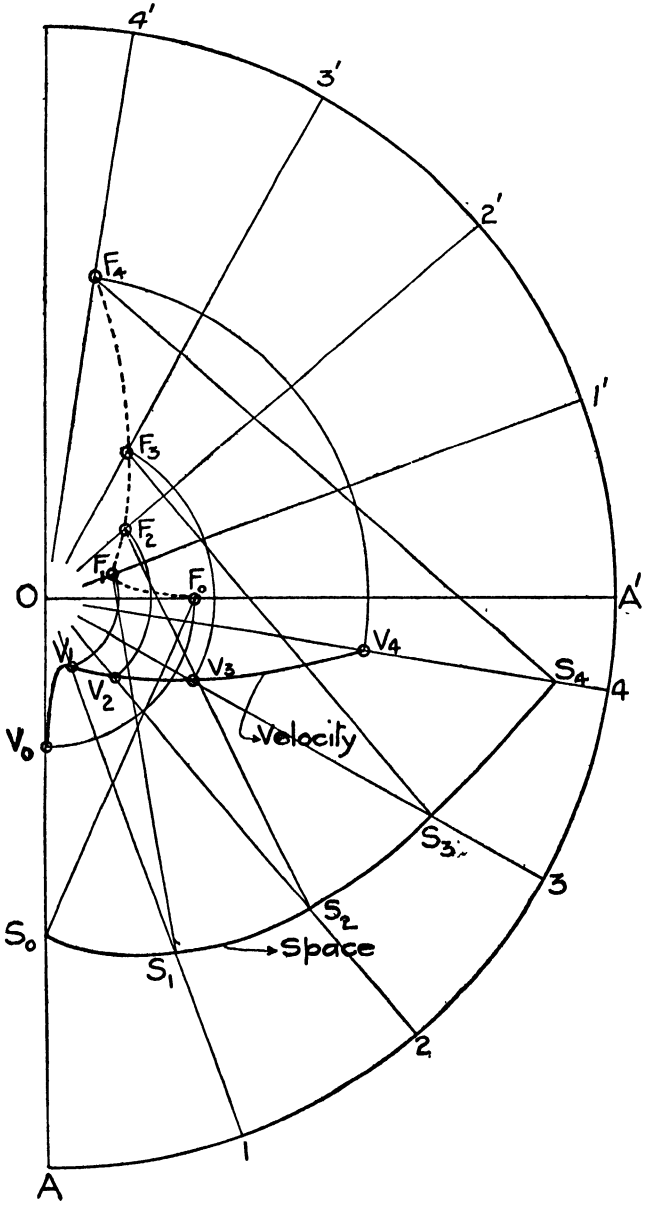


FIG. 31.—Polar Velocity Curve.

we can then get at once the lines at right angles to $O\,S_0$, $O\,S$, etc., and by reading from A′ we can also use the curve $F_0$, $F_1$, etc., as the polar velocity curve. Now the curves $S_0$, $S_1$, etc., and $F_0$, $F_1$, etc., bear to each other a relation similar to that which we have referred to as sum curve and slope curve in the case of rectangular or Cartesian ordinates; in the present case we will, by analogy, call the curve $F_0$, $F_1$, etc., the *polar slope curve* of the curve $S_0$, $S_1$, and the curve $S_0$, $S_1$ we shall call the *polar sum curve* of the curve $F_0$, $F_1$, etc.

**Extension to Polar Acceleration Curve.**—Acceleration is defined as the rate of change of velocity, *i.e.* if the velocity increases by an amount $\delta v$ during a short time interval $\delta t$, the mean acceleration is given by the relative

$$a = \frac{\text{increase of velocity}}{\text{increase of time}} = \frac{\delta v}{\delta t}.$$

There is, therefore, exactly the same kind of relation between the acceleration and velocity at any point as between the velocity and space, so that the same construction as is employed to find the velocity at any point from the polar space curve will give the acceleration when applied to the polar velocity diagram. This enables us to enunciate the following rules:

(1) *The polar slope curve of a polar space curve gives the polar velocity curve, and the polar slope curve of the polar velocity curve gives the polar acceleration curve.*

(2) *The polar sum curve of a polar acceleration curve gives the polar velocity curve, and the polar sum curve of the polar velocity curve gives the polar space curve.*

SCALES.—Suppose the space scale is $1'' = x$ ft., and the time scale is $90° = y$ secs., then the velocity scale will be $1'' = \frac{\pi x}{2y}$ ft. per sec. If, for instance, the space scale is $1'' = 4$ ft., and the time scale is $90° = 10$ secs., the velocity scale will be $1'' = \frac{3{\cdot}1416 \times 4}{20} = 0{\cdot}628$ ft. per sec. Similarly,

if the velocity scale is $1'' = z$ ft. per sec., and the time scale is $90° = y$ secs., the space scale will be $1'' = \frac{2yz}{\pi}$ ft., so that if the velocity scale is $1'' = 2$ ft. per sec., and the time scale is, as before, $90° = 10$ secs., the space scale will be $1'' = \frac{2 \times 2 \times 10}{\pi} \doteq 12{\cdot}7$ ft. Extending this to the acceleration scale, we shall have in the above notation:

$$\text{Acceleration scale } 1'' = \frac{\pi z}{2y} = \frac{\pi^2 xz}{4y^2} = \frac{10xz}{4y^2} \text{ nearly.}$$

Similarly, if the acceleration scale is $1'' = u$ ft. per sec., the velocity scale is $1'' = \frac{2yu}{\pi}$ ft. per sec., and the space scale is $1'' = \frac{4y^2u}{\pi^2}$ ft., $= \frac{4y^2u}{10}$ ft. nearly.

Taking, therefore, an acceleration scale of $1'' = 2$ ft. per sec. per sec., and a time scale, as before, $90° = 10$ secs., we have

$$\text{Velocity scale } 1'' = \frac{2 \times 10 \times 2}{\pi} = 12{\cdot}7 \text{ ft. per sec.}$$

$$\text{Space scale } 1'' = \frac{4 \times 100 \times 2}{10} = 80 \text{ ft.}$$

**Application to Simple Harmonic Motion.**—We will now consider the application of the above polar diagrams to the case of simple harmonic motion. It is well known that the curve giving spaces of simple harmonic motions against times in rectangular co-ordinates is a curve of sines. This is expressed in most general form in mathematical language:

$$s = d \sin (bt + c), \quad \ldots\ldots\ldots\ldots \quad (7)$$

where $b$, $c$, and $d$ are constants.

If we count our times and spaces such that $t = 0$ when $s = 0$, this gives $0 = d \sin (0 + c) = d \sin c$, but $\sin 0 = 0$; $\therefore c = 0$. This quantity $c$ may be called the *lead.* We can, therefore, always reduce our formula to the form

$$s = d \sin bt \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

We shall indicate later the effect that $c$ has upon the various curves.

Now take a circle A B E F, Fig. 32, and upon the vertical diameter B F draw two circles B S O, O G F, each with a collinear radius of the main circle for its diameter.

Now take any point S upon one of these circles and join O S, producing it to meet the main circle in C, and draw C D perpendicular to O A and join S B.

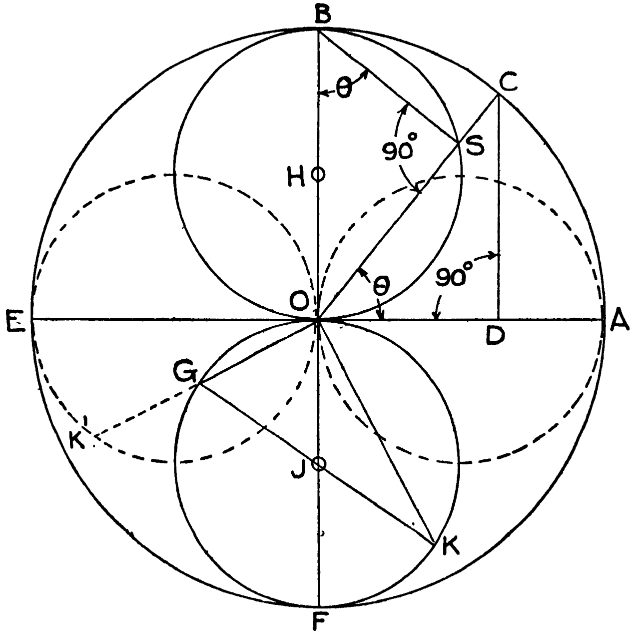


FIG. 32.—Application to Simple Harmonic Motion.

Then $\angle$ B S O, being that in a semicircle, is 90°.

$$\therefore \angle \mathrm{OBS} = 90° - \angle \mathrm{BOS} = \theta.$$

Now $\quad \sin \theta = \frac{\mathrm{CD}}{\mathrm{OC}}$ from the $\triangle$ C O D,

and $\quad \sin \theta = \frac{\mathrm{OS}}{\mathrm{OB}}$ from the $\triangle$ B O S.

$$\therefore \text{ Since } OA = OB,$$
$$OS = OA \sin\theta;$$

but if $\theta$ represents times, we can choose the scales such that $\theta = bt$, and we may take $OA = d$.

$$\therefore\ OS = d \sin bt.$$

$\therefore$ $OS = s$, the displacement in the simple harmonic motion.

We can prove that the above relation holds for any point that we may take upon the two circles, so that we see that the polar space curve for simple harmonic motion consists of two equal circles tangential to, and on either side of, the line of zero time.

**Polar Velocity and Acceleration Curves.**—Take any point G on the polar space curve and apply to it the construction for obtaining the velocity. The normal to the circle is the radius, so draw the radius G J and produce it to meet the line perpendicular to O G in K. Then O K gives the velocity; but since the angle in a semicircle is a right angle, the point K must be on the opposite end of the diameter to G. The circles, therefore, are also the polar velocity curves twisted through 90°, because we have already shown that points such as K, when joined up, are of exactly the same shape as the polar velocity curves turned through 90°.

The polar velocity curve for simple harmonic motion, therefore, consists of two circles, as shown in dotted lines in Fig. 32.

A similar consideration shows that the polar acceleration curve will also be two circles at right angles to the velocity circles and coinciding, therefore, with the space circles, but displaced by 180°, so that the point B on the space circle corresponds to the point F on the acceleration circle. The last-mentioned relation could also be deduced from the definition of simple harmonic motion which is often adopted when first dealing with the subject, namely, that the motion

is such that the resultant force (which is proportional to the acceleration) is proportional to the displacement from the centre of motion, and acts in a direction opposite to the motion.

By suitably choosing the acceleration and space scales, the lengths representing the two quantities thus become equal and opposite.

**Application to Valve Diagrams ; Zeuner's Diagram.**—The above polar space curve for simple harmonic motion is of great value in problems concerning the design of slide valves for steam-engines, and in that connection is called Zeuner's valve diagram.

The motion of the valve of a steam-engine, as we shall show later, approaches simple harmonic motion very nearly, the difference becoming less as the length of the eccentric increases with reference to the throw of the eccentric.

Referring to Fig. 33, O A represents the crank and A C the connecting-rod for the piston, and O B represents the eccentric radius and B F the eccentric rod for the slide valve. Then the points F and C move in approximately simple harmonic motion. The eccentric O B is in advance of the crank by the angle A O B $= 90 + \theta$.

The crank-pin moves with uniform motion in a circle, so that the angle $\alpha$, which represents the position of the crank, also represents the time in moving from C to E.

Draw a circle with diameter K J equal to the valve travel, and draw the diameter P Q of this circle at angle $\theta$ to the vertical diameter L M, as shown ; on this diameter draw the circle O R P, O T N.

Referring back to Fig. 32, it will be noticed that when times are measured from the centre of motion the circles are drawn upon a vertical diameter. Now when the valve is at the centre of its travel, the crank is at the point A′, which corresponds to the point U upon the polar diagram ; so that if we measure our times from the line O U, the polar space diagram for the valve will be at right angles to this,

and will thus be the two circles drawn. Now draw any radius, O Q cutting the polar curve in R. Q U represents the time from the centre of the valve travel, and since the crank position in time at its end E is the angle $\theta$ behind

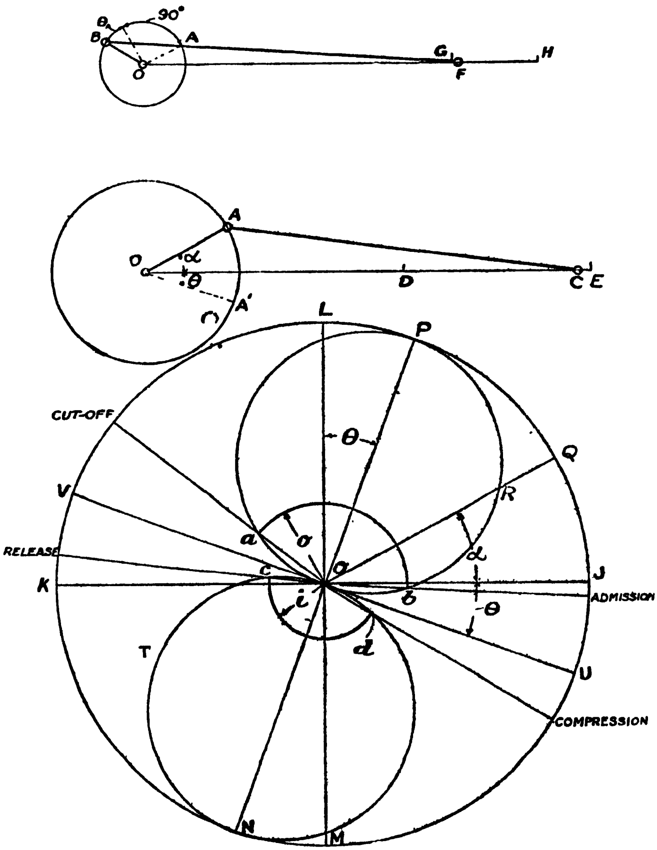


FIG. 33.—Application to Zeuner's Valve Diagram.

the valve at its centre, the $\angle$ Q O J gives the corresponding crank angle, so that for the crank angle $\alpha$ the movement of the valve from its central position is O R. The arrangement of the simple slide valve is shown diagrammatically in Fig. 34. The slide valve A is shown in its central position ;

as it moves to the right it will uncover the port $I_1$, which connects with the left-hand end of the cylinder, and thus open this port to the steam in the space S. The port $I_2$, connecting with the other end of the cylinder, will also be brought into communication with the exhaust port E by means of the gap T. Movement in the opposite direction will open the port $I_2$ to steam and open the port $I_1$ to exhaust. The interval between the closing of the ports to steam and the open to exhaust, and *vice versa*, is governed by the overlapping lengths $i$ and $o$.

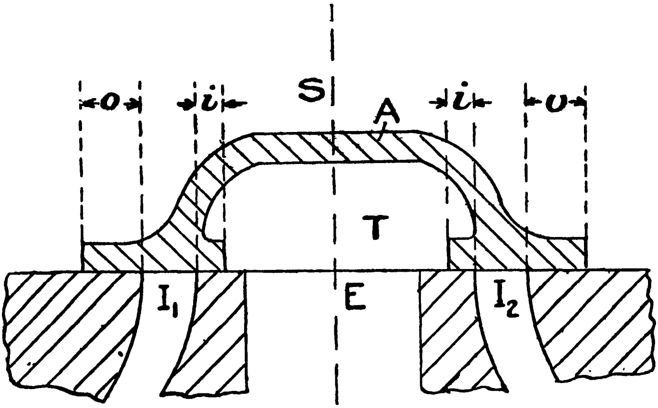


Fig. 34.—Simple Slide Valve.

The quantity $o$ is called the *outside lap* and gives the distance that the valve must move to the right from its central position before the inlet port $I_1$ begins to open to the steam, and the quantity $i$, called the *inside lap*, represents the distance that the valve must move to the right from its central position before the port $I_1$ communicates with the exhaust port E.

On the Zeuner circles draw arcs of radii $o$, $i$ respectively, with centre O cutting the circles in $a$, $b$ and $c$, $d$. The inlet port begins to open when the valve movement is equal to $o$, and O $b$ is the valve movement in the given position, so that by joining O $b$ and producing we get the angle of the

crank circle at which steam is admitted, or the "admission angle," and the valve closes again when the movement gets equal to $o$ again, so that O $a$ produced gives the angle of "cut-off." Similarly, O $c$ and O $d$ produced give the angles at which the exhaust opens and closes, these angles being called "release" and "compression" respectively. All these angles are measured from the line O J.

We can now see the effect of the quantity $c$ in the general equation (1) of the simple harmonic motion; it is to swing round the diameter upon which the polar circles are drawn by an angle equal to $c$.

**The Combined Effect of Two Simple Harmonic Motions.**—Suppose that two simple harmonic motions of the same time period and in the same direction act at the same time upon a body. Let the circles 1, 2, Fig. 35, be the polar space curves for the two separate motions, the angle $\beta$ between their diameters representing the difference in "leads" of the motion. Sometimes this is called the difference in "phase."

Then, if any radius vector O C D be drawn, O D represents the distance from the centre of motion of the first motion, and O C represents the corresponding distance for the second motion, so that the relative distance between the two motions is equal to C D, *i.e.* C D represents the relative motion of the point. Now join A B and draw O E parallel to A B and B E′ parallel to O A.

Produce E′O to E, making O E = O E′, and upon O E and O E′ as diameters draw two circles Z as shown in dotted lines.

Then these two circles will be the polar space curves for the relative effect of the two simple harmonic motions.

PROOF.—Let O D cut the circles 3 in F. Join E F and A D, and draw A G parallel to O D, and B G parallel to A D.

Consider the $\Delta$s B A G and E O F. The angle E F O is a right angle, being the angle in a semicircle; the angle A D O is a right angle for the same reason, and because

A D C G is a parallelogram, the angle A G B will also be a right angle.

Therefore B G and E F are both perpendicular to O D and are thus parallel.

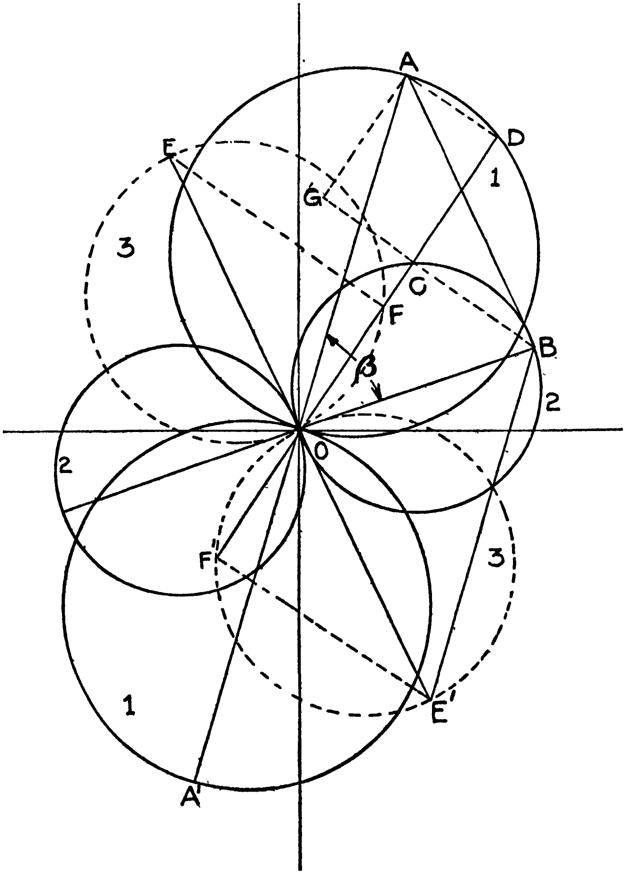


FIG. 35.—Polar Diagram for Two Simple Harmonic Motions.

The two $\Delta$s, therefore, have their sides mutually parallel and are thus similar.

Also O E = O E′ = A B (because O E′B A is a parallelogram). Therefore the two $\Delta$s are equal to each other in all respects, so that O F = A G = C D.

We see, therefore, that *the relative effect of two simple harmonic motions is itself a simple harmonic motion*, the polar diagrams for which can be obtained by the above construction.

This result is very useful in practice in the case of Meyer's expansion valve for steam-engines; in this device a second eccentric set in advance of the first is used to operate a second slide valve to control the cut-off of the steam. The above construction enables us to find the throw and setting of a single eccentric for one slide valve which will have the same effect as the combined actions of the two. The second slide valve of Meyer's device is made adjustable by right- and left-hand screws to enable the cut-off of the steam to be varied while the engine is running.

**The Case in which the Two Motions Agree in Phase.**—If the two motions synchronise, *i.e.* agree in phase so that the angle $\beta$, Fig. 35, is zero, the above construction cannot be carried out. In this case we proceed as follows:

The polar space curves for the two simple harmonic motions will be the two pairs of circles 1, 2, Fig. 36, whose diameters O A, O B are upon the same straight line. Draw a circle upon O E as diameter, making O E = A B; then this circle, shown in dotted lines, and its companion on the opposite side will be the polar space curves for the two motions.

To prove this draw any radius vector O D cutting the three circles 1, 2, 3 in D, C, F respectively. Draw B G parallel to C D. Then the angle O D A is the angle in a semicircle and is thus a right angle; the angle O F E is a right angle for the same reason, so that E F is parallel to A G.

The $\Delta$s O E F, B A G have their sides mutually parallel, so that they are similar. Moreover, by the construction O E = B A, so that the triangles are equal in all respects.

$$\therefore \text{O F} = \text{C D}.$$

But C D is the relative movement of the point under the two motions.

∴ O F is the relative movement, so that the circles drawn in dotted lines are the polar space curves for the relative motion.

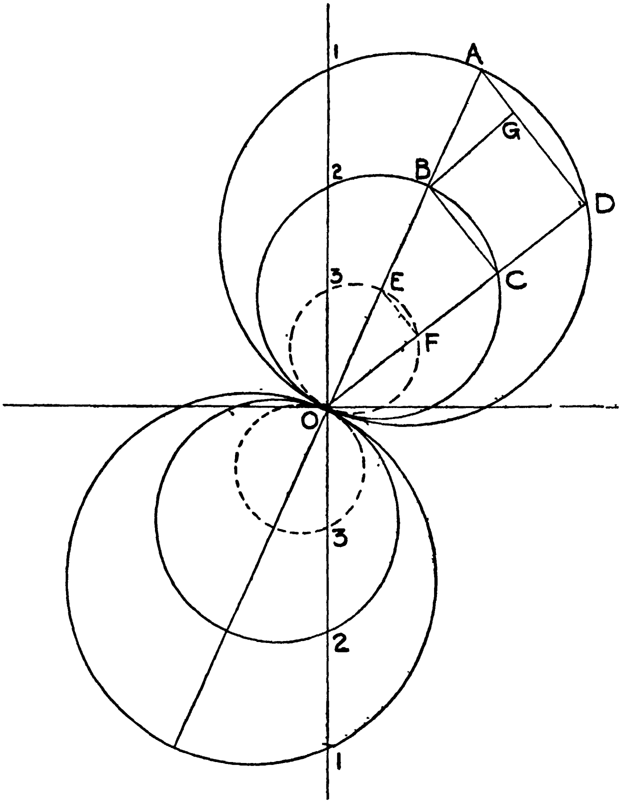


Fig. 36.—Polar Diagram for Two Simple Harmonic Motions in Phase.

In the case, therefore, of two synchronous simple harmonic motions, the relative effect is also a simple harmonic motion synchronous with the other two and of amplitude equal to the difference between the amplitudes of the separate motions.

**Polar Sum Curve Construction.**—We have considered the construction which enables us for polar diagrams to find the curve corresponding to the slope curve of the rectangular co-ordinate diagrams, and we now come to the inverse construction for giving the curve which, by analogy, we will call the *polar sum curve.*

Let X, Fig. 37, be the polar curve whose polar sum curve is required. Draw a number of radius vectors O 1, O 2, etc., at a convenient small angular distance apart, preferably equal. For convenience of drawing, these radius vectors can be set out as shown upon a circle with centre at O. Now draw the corresponding mid-angular vectors, cutting the curve X at points $1a$, $2a$, $3a$, etc.

From the extremity B′ of the diameter to the circle mark off points 1′, 2′, 3′, etc., as shown, the distance B′1′ being equal to A 1, etc.

Before we can draw the polar sum curve we must know the value at the beginning of the period of the quantity which the sum curve is to represent. To the scale which we shall presently determine set out O $a$ to represent this initial value, and with centre $1a$ strike an arc $a\,b$ to cut the radius vector O 1′ in $b$. Then with centre $2a$ draw an arc $b\,c$ across the next sector, and so on, the last piece, $f\,g$, being drawn across the sixth sector with centre (Fig. 37) $6a$. The resulting curve, $a\,b\,c\,.\,.\,.\,g$, is the polar sum curve required and can be transferred, as shown in dotted lines, on to the same vectors of the original curve. In general, however, this transfer is not necessary.

The proof of this construction is as follows: Consider any elemental arc—say $d\,e$—of the polar sum curve. $4a\,h$ is the radius of this arc and is so normal to it. Moreover, O $4a$ is at right angles to O $h$, so that O $4a$ represents the polar slope at $h$ of the curve $a\,b\,c\,.\,.\,.\,g$, in accordance with the construction proved with reference to Fig. 35.

Scales.—Let the original curve X be drawn to an angular scale $90° = y$ units, and a radius vector scale $1'' = x$ units;

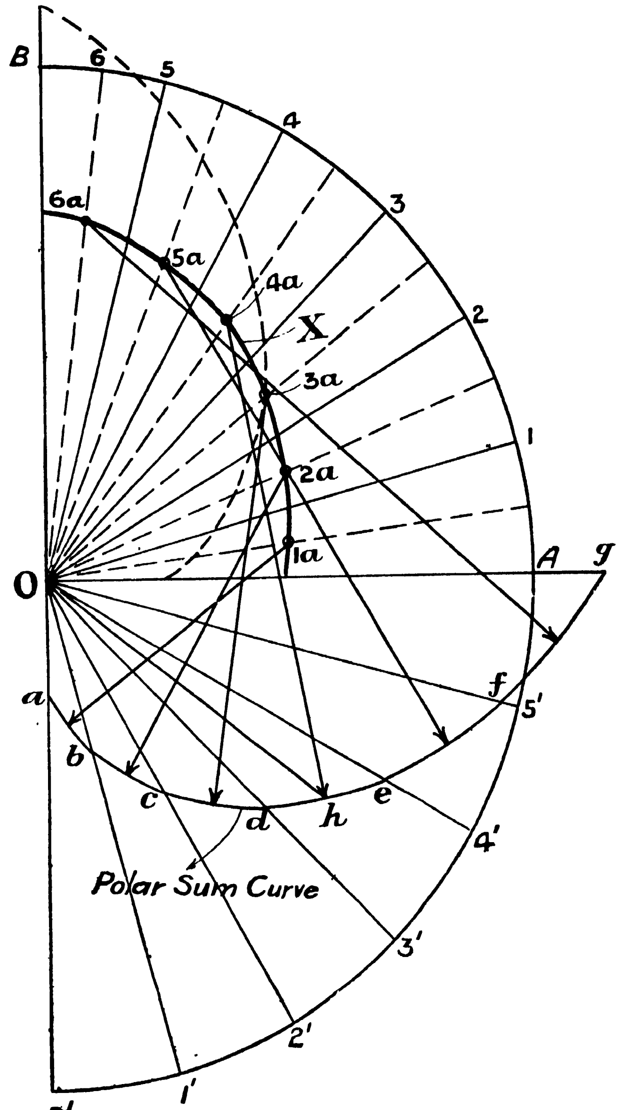


FIG. 37.

then the radius vector scale of the polar sum curve will be $1'' = \frac{2xy}{\pi}$ units.

**Special Case of Constant Acceleration.**—If the acceleration of a point is constant, the polar acceleration diagram will be a circle with centre at the origin, as shown in Fig. 38. But from the definition of acceleration it follows that for constant acceleration the gain in velocity is equal to

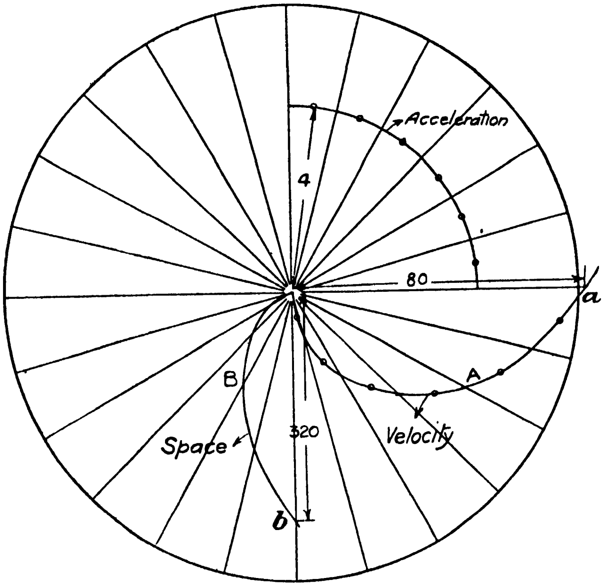


FIG. 38.

equal time intervals; so that the polar velocity diagram will be such that for equal angles (time intervals) the increase of length of the radius vectors (velocities) is equal. But the curve that has this property is known as the *spiral of Archimedes*, so that we get the result that the polar sum curve of a circle whose centre is at the origin is a spiral of Archimedes.

In Fig. 38 the construction previously described for the

polar sum curve has been carried out for the case of a body starting from rest with an acceleration of 4 ft. per sec. per sec. originally drawn to a scale of $1 = 2$ ft. per sec. per sec. and a time scale of $90° = 20$ secs.

The curve A is the polar curve of velocities, and measurement shows $O a = 3{\cdot}14$ in. Now by our formula above for scales, the velocity scale will be

$$1'' = \frac{2xy}{\pi} = \frac{2 \times 2 \times 20}{\pi} = \frac{80}{\pi} = 25{\cdot}5 \text{ ft. per sec.,}$$

so that the velocity after 20 secs. $= O a = 3{\cdot}14 \times 25{\cdot}5 = 80$ ft. per sec.

Repeating the polar sum curve construction for the curve A, we get the curve B, which is the polar space curve. The polar space scale will be $1'' = \frac{2}{\pi} \times \frac{80}{\pi} \times 20 = \frac{3{,}200}{\pi^2} = 320$ ft. nearly. On scaling off the distance $O b$ it was found to be $2{\cdot}5$ in., so that the space covered after 20 secs. will be $2{\cdot}5 \times 320 = 800$ ft.

These figures will be found to be in accordance with the formulæ which are obtained for uniform acceleration when considering diagrams upon a straight base.

In the above diagrams we have shown the curves continued in one quadrant only, but that is only to keep the diagrams as free from confusion as possible. There is no reason why the curves should not continue indefinitely, the time positions repeating after one revolution, but in practice it is usually preferable to restrict the diagram within a quadrant or a semicircle. One further point may be mentioned before passing on to further applications of these curves, viz. that the polar slope curve for a circle concentric with the origin is the origin, *i.e.* the polar slope curve becomes a point. This will be seen quite clearly from the case of space and velocity diagrams. If the polar space curve is a circle, the body remains at the same distance from the origin, *i.e.* the body is stationary; the velocity,

therefore, at any point is zero, or the polar velocity diagram is a point coinciding with the origin.

**Application of Polar Space Diagrams to Cams.**—There are some interesting applications of polar diagrams

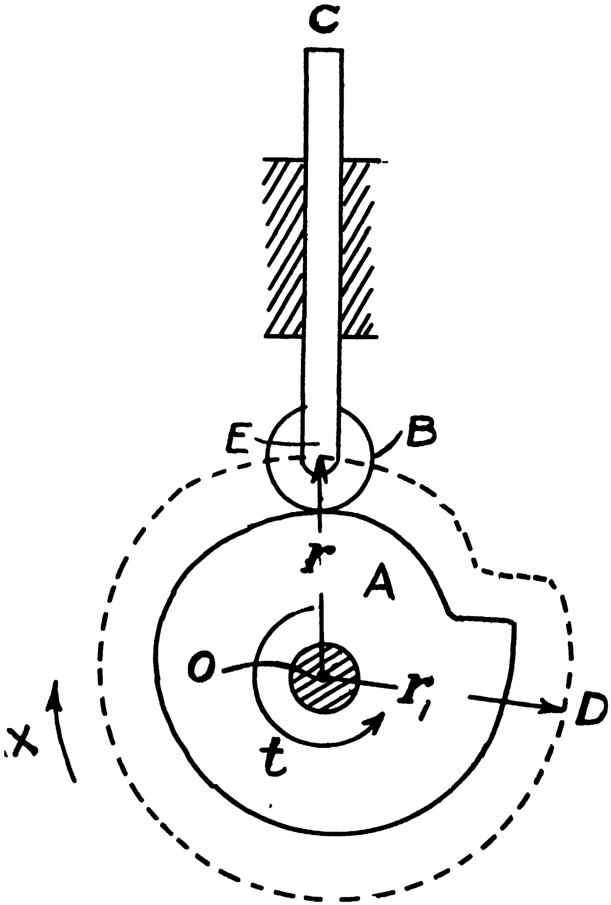


FIG. 39.

to the design of cams. We will restrict the term "cam" to a shaped body A, Fig. 39, which rotates uniformly, and upon its periphery acts upon a body C, usually provided

with an anti-friction roller B, the function of the cam being to transmit a variable (and usually intermittent) reciprocating motion to the body C for causing some operative portion of the movement of a machine. Such cams occur in nearly every machine where comparatively small intermittent movements of some parts are required. In practice the cam takes the shape shown in full lines, but its theoretical shape is that shown in dotted lines, this being the shape required to give the requisite motion to the part C irrespective of the roller.

To get a clear idea of the action of the cam, suppose that it rotates in the direction of the arrow X, and that after a time $t$ the point D is at the centre of the roller. In this time $t$ the body C has risen by the amount $(r_1 - r)$—that is, the radii of the cam represent the positions of the body C; in other words, if the direction of motion of the body passes through the axis of the cam, *the shape of the cam is the polar space diagram for the motion which the cam is required to give.*

As a first example take the case of a cam which is required to give a uniform rise of $2\frac{1}{2}$ in. during the first half of the revolution, then to remain stationary during the next quarter of a revolution, and finally to drop uniformly during the last quarter of a revolution. Take the minimum radius of the cam as $1\frac{1}{2}$ in. and the roller as 1 in. in diameter.

Set out O A, Fig. 40, equal to $1\frac{1}{2} + \frac{1}{2} = 2$ in., so that if O represents the axis of the cam, A represents the axis of the roller at the narrowest portion of the cam. The roller has to rise uniformly for the first half-revolution, and we have shown that for uniform velocity the polar space curve is a spiral of Archimedes, so set out a spiral of Archimedes A E C, O C being equal to O A + $2\frac{1}{2}$ in. = $4\frac{1}{2}$ in. This is done most conveniently in practice by taking a number of equally spaced radius vectors—say ten—and making each increase proportionately in length, *i.e.* if ten are used, each is $2{\cdot}5 \div 10 = 0{\cdot}25$ in. longer than the previous

one. It could, however, be done, if preferred, by the polar sum curve construction by calculating the uniform velocity and drawing the circle which is the corresponding polar velocity curve.

For the next quarter-revolution the roller has to keep stationary, and the polar space curve for this is a circular arc, so draw a circular arc C D. The roller has then to fall uniformly for the remaining quarter of a revolution, so draw a spiral of Archimedes D A, the completed dotted

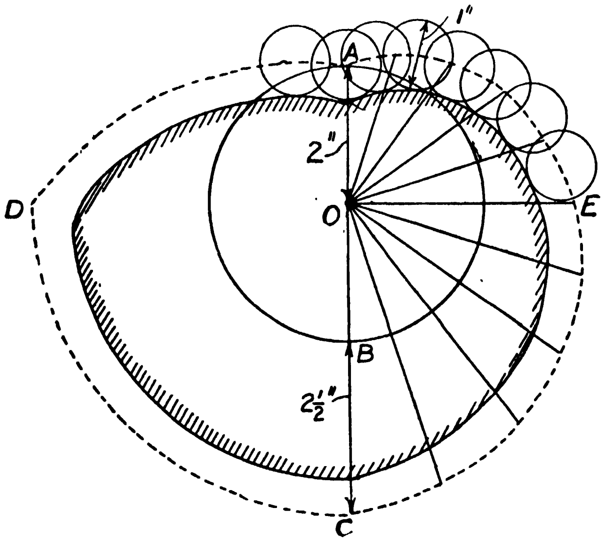


FIG. 40.

curve being the path of the centre of the roller to give the motion required.

Now go round the dotted curve and draw circles of 1 in. diameter close together with centres on the curve, and draw the curve to touch all these circles, the resulting curve or envelope giving the shape of curve required.

**Shape of Cam to Give Simple Harmonic Motion.**— We obtain some rather interesting results by considering the application of the above rules to the determination of the

shape of a cam to give a simple harmonic motion to the centre of the roller.

Let O, Fig. 41, be the cam axis, and let X be the axis of the roller C in its highest position. We have shown

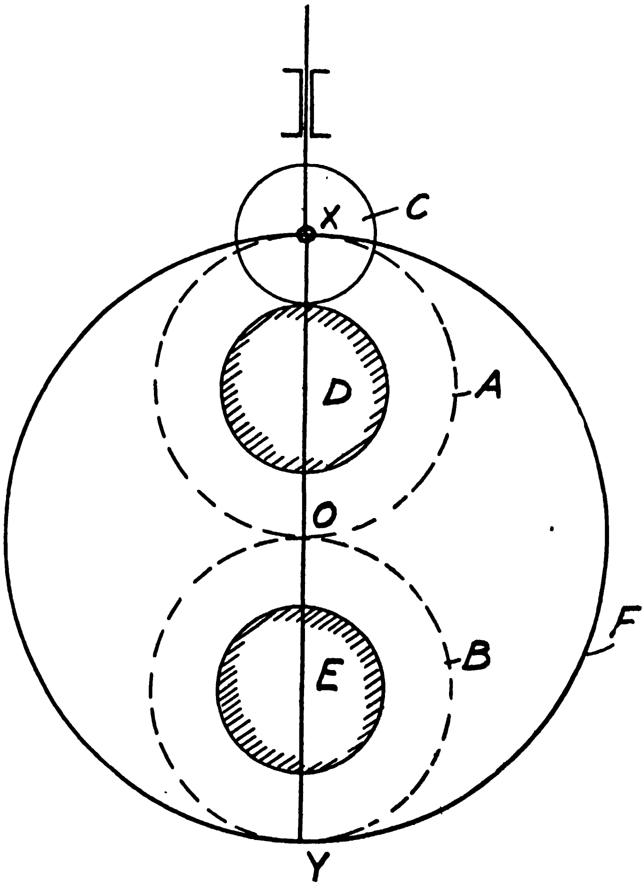


Fig. 41.

already that the polar curve for simple harmonic motion is a pair of circles with a common diameter, and externally tangential to each other, each passing through the origin; therefore draw these circles A, B, the distance X Y being

the total movement to be imparted to the cam. If we go round these circles as before and draw the roller in various positions round them, we shall get the two circles D, E, which are the shapes of the cam to give the motion required. These circles in practice could be made in the form of discs projecting from a main disc F. In practice, however, this cam would not be satisfactory, because when the discs D, E become horizontal the roller would drop right through. If some means were adopted for arresting the roller momentarily when its centre coincides with the centre O, it would not drop further ; on the other hand, it would not necessarily rise or fall, but might rotate in contact with the circles in the same position, unless its weight pressed it out of such position. Mechanisms possessing these properties are said to have "*change points.*" Even if these difficulties were overcome, the other side of the roller D would prevent downward movement, and it will be noted that the second disc E never comes into action. This disc could be made to come into action by causing the roller to start lifting and following up on the disc E after the centre reaches the point O ; the motion, however, would not then be truly simply harmonic, but would consist of a number of repeated motions each of which might be described as part of a simple harmonic motion. The motion is analogous to the effect produced by a two-section commutator upon a dynamo, the resulting current consisting of a series of variable currents which do not alternate in sign. This theoretical form of cam, therefore, is useless in practice. The point is similar to interference which occurs in some internal gears, as explained in Mr. G. T. White's *Toothed Gearing* (Scott Greenwood and Son).

**Polar Curves of Effort with Reference to Space.**—If we plot a polar diagram in which the radius vector represents effort or force (which is proportional to acceleration if the effort is the resultant effort), and the angle represents the distance moved by the body, the resulting curve will

be a polar effort curve, and its polar sum curve will be the polar work curve, from which the velocity can be calculated by the formulæ commonly explained with reference to work curves upon a straight base. This relation holds because the rate of increase of work is equal to the effort, and the polar slope curve represents the rate of increase of its original curve; the polar effort curve is, therefore, the polar slope curve of the polar work curve, and, conversely, the polar work curve is the polar sum curve of the polar effort curve. These curves are seldom used in practice, and so are not fully treated here. The following construction is, however, interesting, and may be useful in some cases.

*To Draw a Polar Curve of Effort from a Polar Curve of Action.*—Let X A Y, Fig. 42, represent a polar curve of

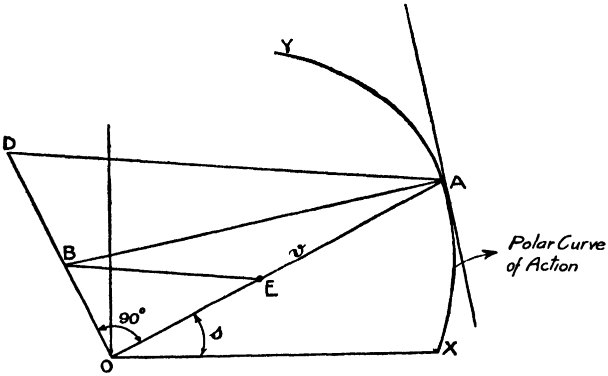


FIG. 42.

action; that is to say, the angle X O A represents the distance $s$ moved through from some reference point, and O A represents the velocity at this distance.

It is well known that the relation between the acceleration and the velocity can be expressed by $a = \frac{v \cdot \delta v}{\delta s}$. (See p. 41.)

We have also pointed out that acceleration is proportional to effort, so that if we have a graphical construction for

finding the acceleration, we can get the effort at once by a change of scale.

Now draw A B normal to the polar curve of action, and draw O B at right angles to O A. Produce O B to D, making O D equal to some convenient polar distance $p$, and join A D and draw B E parallel to it.

Then O E will represent the acceleration to a scale to be explained later, so that E is a point upon the polar curve of effort.

PROOF.—Referring back to Fig. 31, we notice that the above construction as far as the determination of the point B is the same as that for the " polar slope curve."

In connection with that construction we showed that, using the symbols of the present case,

$$\mathrm{O\,B} = \frac{\delta v}{\delta s};$$

$$\therefore\ a = \frac{v\,.\,\delta v}{\delta s} = \mathrm{O\,B} \times \mathrm{O\,A} \qquad \ldots\ldots\ldots\ldots (9)$$

Now B E is parallel to D A.

$$\therefore\ \frac{\mathrm{O\,B}}{\mathrm{O\,D}} = \frac{\mathrm{O\,E}}{\mathrm{O\,A}},$$

*i.e.*
$$\mathrm{O\,E} = \frac{\mathrm{O\,B} \times \mathrm{O\,A}}{\mathrm{O\,D}} = \frac{a}{p}.$$

$\therefore$ O E represents the acceleration $a$, since $p$ is a fixed amount.

SCALES.—Let the velocity scale be $1'' = x$ ft. per sec., and the space scale $90° = y$ ft., and let the polar distance be $p$ actual inches; then the scale of O B will be $1'' = \frac{\pi x}{2y}$, so that the scale of O B $\times$ O A $= a$ will be $1'' = \frac{\pi x^2}{2y}$, and the scale of acceleration will be $1'' = \frac{p \pi x^2}{2y}$.

Now $p$ may be any convenient distance; if we take $p = \frac{10}{\pi}$ or a multiple thereof, the scale of acceleration will be a convenient one.

In applying this construction to the determination of a number of points to get the polar effort curve, a circle can be drawn with centre O and radius $p$, and all the points corresponding to D will be upon this circle.

EXAMPLE.—Suppose that we are required to design a plate cam for a cam-shaft of diameter D, the minimum thickness of the cam to be $t$ and the diameter of the roller $d$.

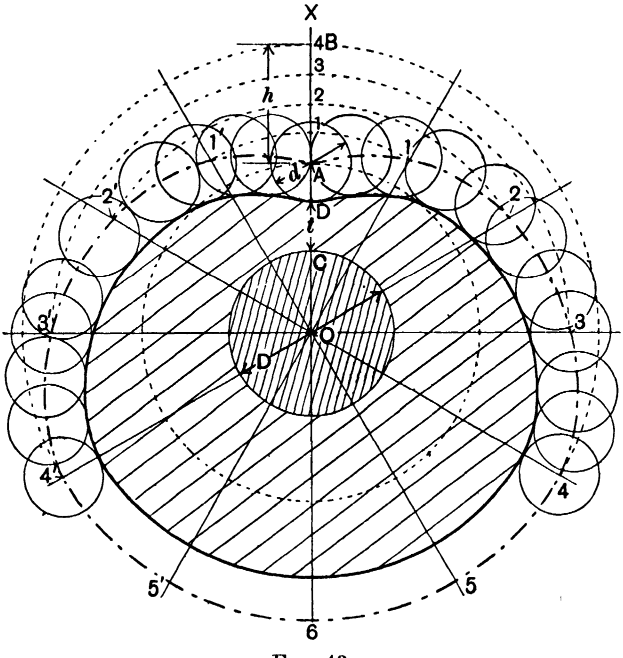


FIG. 43.

The cam is required to move the slide uniformly upward through a height $h$ during one-third of a revolution, then to remain stationary for another third of a revolution, and finally to fall uniformly during the remainder of the revolution.

First draw with centre O a circle of diameter D, Fig. 43, to represent the cam-shaft, and draw the line O X, along which the roller has to reciprocate. Next make C D equal to $t$,

the minimum thickness of the cam, and find the centre A of the roller in its lowest position. A B is next set up equal to the lift $h$, and A B is divided up into a convenient number of equal parts, say four. A number of equally spaced radial lines O 1, O 2, etc., are then drawn, twelve being taken as a convenient and sufficient number.

In moving through one-third of a turn, *i.e.* from O A to O 4, we have to rise a height $h$ and have to do so uniformly; we therefore make O 1 = O 1 and O 2 = O 2; O 3 = O 3 and O 4 = O 4, as indicated by the circular arcs shown in dotted lines. During the next one-third of a revolution the roller has to remain stationary, so that we draw an arc with centre O from 4 to 4′, and since the roller has to fall uniformly during the remainder of the revolution we repeat the construction for the points 1, 2, 3, 4 to get the points 1′, 2′, 3′, 4′.

By joining up the points thus obtained we derive the curve shown in chain dotted lines, this curve being that for the centre of the roller. We then go round the curve and draw the roller circle all round it to give the effect of the roller running along such a curved path. By drawing a line to touch the roller on the inner side in all its positions we get a curve which a mathematician would call an envelope, and which gives us the shape of the cam required.

# CHAPTER V

## DIAGRAMS FOR VELOCITY CHANGING IN DIRECTION; THE HODOGRAPH

We have considered so far only the case of motion in a straight line and have taken into consideration only changes in magnitude of the velocity; but we may also have change in direction, with or without change in magnitude as well. The case of a body moving with constant velocity in a circle is an example in which the magnitude of the velocity is constant, but the direction is constantly changing.

**Combination of Velocities.**—The actual velocity possessed by a point may be the combination of two or more velocities, and as velocities are vector quantities they are added together in exactly the same way as forces, *i.e.* by the law of vector addition. Suppose, for instance, that we are standing at one end A, Fig. 44, of a railway carriage moving

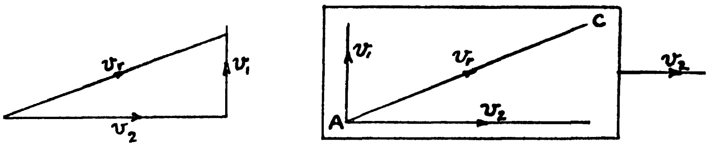


FIG. 44.—Combination of Velocities.

with a velocity $v_2$, and that we walk across the carriage with a velocity $v_1$; then our actual velocity will be the combination of the velocity $v_2$ of the train itself and of our own velocity $v_1$, *i.e.* $v_r$ in the direction A C by the law of vector addition.

A good familiar example in which a body has a velocity compounded of two velocities is to be obtained from the case of a wheel rolling along the ground. Any point on the wheel is moving around the axle, and the axle is at the same time moving along parallel to the ground, so that each point upon the wheel is actually describing a curved path—indicated in dotted lines in Fig. 45. This curved path is called

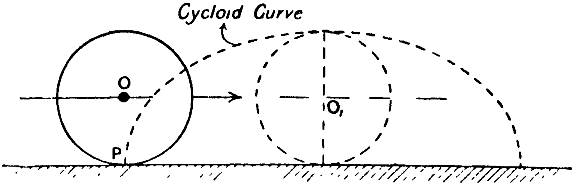


FIG. 45.—Wheel Rolling along the Ground.

the *cycloid* and is used in considering gear-teeth. This curve can be drawn by rolling a half-crown along a ruler and resting a pencil against the edge of the coin.

**Change of Velocity.**—Suppose that a point at A, Fig. 46, has a velocity $v_1$ at one instant, and after a certain time it

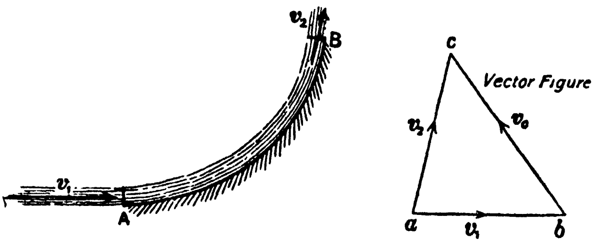


FIG. 46.—Change of Velocity.

is at B and has a velocity $v_2$. Then the change of velocity $v_c$ is defined as the velocity which would have to be compounded or combined with $v_1$ to give $v_2$. That is, $v_2$ is the resultant of $v_1$ and $v_c$, or, expressing this in vector notation, we have

$$v_2 = v_1 \mathbin{+\!\!\!+} v_c .*$$

* The modified *plus* sign $\mathbin{+\!\!\!+}$ indicates vector addition, *i.e.* addition in which direction is considered as well as magnitude.

This problem arises in engineering calculations in considering the impact of water upon the vanes of a water-wheel or turbine, which is dealt with in detail on p. 98.

NUMERICAL EXAMPLE.—*A jet of water moving with a velocity of* 80 *ft. per sec. impinges upon a curved plate and has its direction turned through* 120°, *without altering its magnitude. What is the change in velocity?*

Referring to Fig. 47, we draw $a\,b$ to a suitable scale to represent 80 ft. per sec., and $a\,c$ at 120° to it to represent 80 ft. per sec. also, and then join $b\,c$; then $b\,c$ represents the change of velocity, and if the diagram is drawn to scale $b\,c$ will be found by measurement to give about 138·6 ft. per sec. To find $v_c$ by calculation without actually drawing

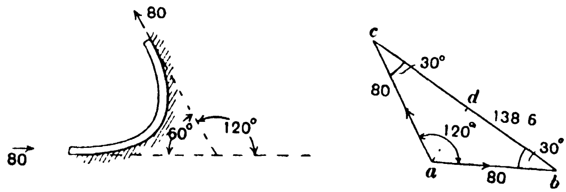


FIG. 47.

the triangle to scale, we draw $a\,d$ perpendicular to $b\,c$; we then note that

$$c\,d = a\,c \cos 30° = 80 \times 0{\cdot}866;$$

$$\therefore\ b\,c = 2\,c\,d = 160 \times 0{\cdot}866 = \underline{138{\cdot}6 \text{ ft. per sec. nearly.}}$$

**Relative Velocity.**—In the ordinary way when we speak of velocities in a certain direction (say four miles an hour in a northerly direction) we leave out of consideration the fact that the earth is not fixed. Since the earth itself is rotating on its axis as well as moving through space at a very high velocity, the actual velocity of any point is the combination of the velocity commonly referred to and that of the earth. We express this by saying that velocities as ordinarily measured are *relative* to the earth.

If we sit at the back of a dog-cart we can easily get the idea that the road is moving away from under us; that is because we regard ourselves as fixed, and therefore relatively the road is moving away from us.

If, again, two trains are standing alongside in a railway station and one starts moving, a person sitting in one train and looking at the other always has some doubts as to which of the trains is moving. Suppose that we are sitting in the train which we will call A, and that the other is B. If B moves we have the sensation of moving in the opposite direction.

Now if two bodies A and B are both moving, *the velocity of* B *relative to* A *is the velocity which* B *would appear to have if* A *were regarded as stationary.*

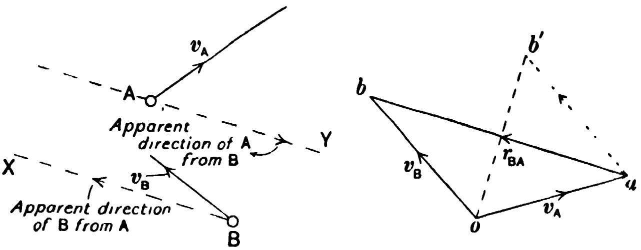


FIG. 48.—Relative Velocities.

Suppose that a point A, Fig. 48, is moving with a velocity $v_A$ with reference to a certain plane in the direction indicated, and that the point B is simultaneously moving with a velocity $v_B$ with reference to the same plane in the direction indicated, A and B being positions of the points at the same instant.

To a convenient scale set out $o a$ parallel to $v_A$ to represent $v_A$ in direction and magnitude, and to the same scale set out $o b$ to represent $v_B$ in direction and magnitude, and join $a b$; then $a b$ is the velocity of B relative to A, *i.e.* $a b$ is the velocity which B appears to have to a person moving with A; it is written $r_{BA}$. Similarly, $b a$ is the velocity of A relative to B.

It will be noted that this construction is different from that employed for finding the resultant of $v_A$ and $v_B$; if the resultant had been required, we should have drawn $a\,b'$ to represent $v_B$ as indicated in dotted lines, and $o\,b'$ would have given the resultant, and is the vector sum.

It will be noticed that $v_B$ is the vector sum of $v_A$ and $r_{BA}$, *i.e.* using the vector notation,

$$v_B = v_A \nrightarrow r_{BA}.$$

Therefore, using $\sim$ to indicate vector difference, we have

$$r_{BA} = v_B \sim v_A.$$

Expressing this in words, we see that *the velocity of* B *relative to* A *is the vector difference between the velocity of* B *and the velocity of* A.

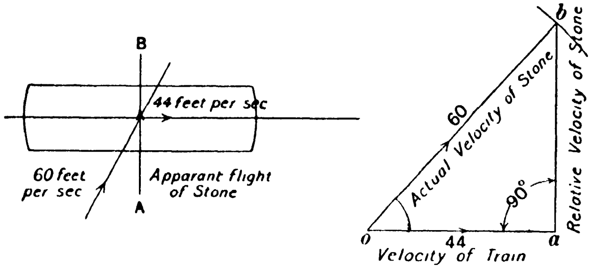


FIG. 49.

The dotted lines B X and A Y, which are each parallel to $a\,b$, are the paths which B and A appear to take from A and B respectively.

NUMERICAL EXAMPLES.—(1) *If a train is running at* 30 *miles an hour, in what direction must a stone be thrown at a velocity of* 60 *ft. per sec. to pass in through one open carriage window and out through the opposite window?*

Referring to Fig. 49, the stone must have a velocity relative to the train in a direction A B, *i.e.* at right angles to the direction of motion of the train.

30 miles an hour = 44 ft. per sec., so let $o\,a$ represent 44 ft. per sec.; draw $a\,b$ at right angles to $o\,a$, and with $o$

centre draw an arc of radius representing 60 ft. per sec. ıtting $a\,b$ in $b$. This determines the point $b$ and completes ıe triangle of velocities. We want to find $\angle\ a\,o\,b$ to obtain ıe direction in which the stone must be thrown.

If we draw to scale we shall find that the $\angle\,a\,o\,b$ is about 3°.

(2) *A ship* A *is steaming due N. at a speed of* 10 *miles an* ›*ur ; when another boat* B *is due W. of* A *and at a distance* 1 *miles from it,* B *starts at a speed of* 10 *miles an hour in a* .*E. direction. What is the least distance apart that* B *will* *tain from* A, *and how long after starting will* B *be at its least* '*stance from* A ?

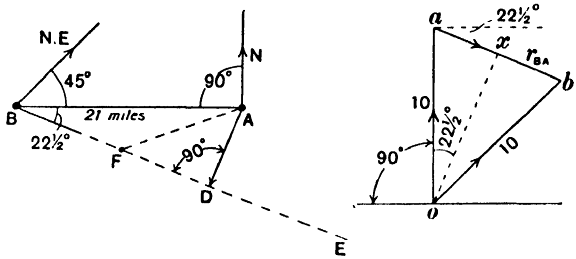


FIG. 50.

Fig. 50 indicates the position of the boats at the first ıstant under consideration.

We first draw the vector figure to obtain the relative elocity of B to A. Draw $o\,b$ in a N.E. direction to represent 0 miles an hour, and $o\,a$ in a N. direction to represent 10 ıiles an hour also. Then $a\,b$ is the velocity of B relative › A, *i.e.* $a\,b$ is the velocity which B appears to have to a erson on A. Now draw B E parallel to $a\,b$; then if A 'ere fixed, B E would be the path taken by the steamer . The boat B, therefore, will appear from A to move long the line B E with a velocity represented by $a\,b$.

If the $\Delta\,o\,a\,b$ be drawn carefully to scale, $a\,b$ will be found › represent 7·65 miles an hour.

If we do not plot to scale we can calculate $a\,b$ as follows: Draw $o\,x$ perpendicular to $a\,b$.

Then $$\frac{a\,x}{o\,a} = \sin 22\tfrac{1}{2}°,$$

*i.e.* $$a\,x = 10 \sin 22\tfrac{1}{2}°;$$

$$\therefore\ a\,b = 2\,a\,x = 20 \sin 22\tfrac{1}{2}°$$
$$= 7{\cdot}654 \text{ miles per hour.}$$

Suppose that after a given time, say one hour, B has arrived at F in its apparent path B E; then B F will be 7·65 miles, and A F will be the distance of B from A at that instant; in other words, the distances from A to various points on B E give the distances apart of the boats at various times.

The least distance apart of the boats will, therefore, be given by A D, where A D is drawn perpendicular to B E. By measurement this should come to 8·04 miles.

By calculation we have

$$\frac{\text{A D}}{\text{A B}} = \sin 22\tfrac{1}{2};$$

$$\therefore\ \text{A D} = 21 \sin 22\tfrac{1}{2} = \underline{8{\cdot}036 \text{ miles.}}$$

**The Hodograph.**—We have considered up to the present only the case in which the motion of a body takes place in a straight line. If the body moves in a curved path, its motion may be in many cases considered most conveniently by means of a polar diagram called the *hodograph*, which is defined as follows:

Let $P_0$, $P_1$ . . . $P_4$, Fig. 51, etc., represent successive points upon the curved path of a body, and let $P_0$ 0, $P_1$ 1, $P_2$ 2, etc., be the tangents to the curve at the various points.

Taking a pole X, draw a vector X O parallel to the tangent to represent the velocity $v_0$ of the body at the point $P_0$ to some convenient scale; then draw X I parallel to $P_1$ 1 to represent the velocity $v_1$ at $P_1$ to the same scale, and so on. Then the curve obtained by joining the points 0, 1, 2 . . . 4, is called the velocity hodograph for the motion.

Now consider the question of acceleration. Acceleration is defined as the rate of change of velocity; in the cases that we have considered up to the present we have only had a change of magnitude of the velocity to consider because the direction is constant. In the general case, however, the change of velocity may consist of a change of direction

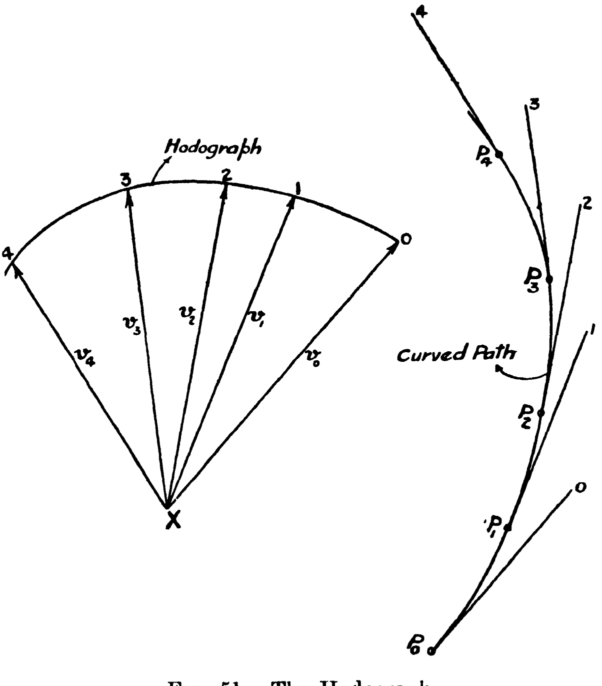


FIG. 51.—The Hodograph.

as well as one of magnitude. In the case under consideration, for instance, the velocity between the points $P_1$, $P_2$ changes from X 1 to X 2, the change in velocity being represented by the vector difference 1, 2. If the distance $P_1$ $P_2$ is very short, and the time taken in traversing it is $\delta t$, we have

$$\text{Acceleration} = a = \frac{1,2}{\delta t}.$$

This means that the acceleration of the body between the points $P_1$ $P_2$ is equal to the velocity with which the hodograph moves across the corresponding period. This gives us the rule that *the velocity in the hodograph is equal to the acceleration in the curved path* ; the acceleration at any point will also be in the direction of the tangent to the hodograph at the point.

If, therefore, we consider the velocity hodograph as a curved path and repeat the construction, the new curve will give accelerations and may be called the acceleration hodograph.

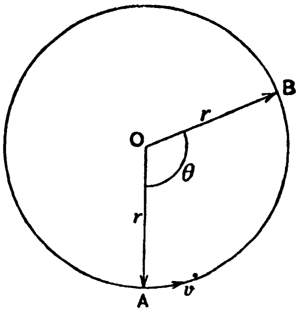


FIG. 52.—Angular Velocity.

**Uniform Motion in a Circle; Angular Velocity.** — Suppose that a point moves with a velocity $v$ in a circle of radius $r$, and that in a time $t$ the point moves through an arc AB, Fig. 52, subtending an angle $\theta$ at the centre of the circle.

Then the angle turned through in a unit time is called the *angular velocity* and is given the letter $\omega$.

Then since arc = angle (in radians) × radius, we have

$$AB = r\theta,$$

and if $t$ is the time taken from A to B,

$$AB = vt;$$

$$\therefore\ r\theta = vt,$$

or

$$v = \frac{r\theta}{t}; \quad \dots\dots\dots\dots \quad (1)$$

but $\dfrac{\theta}{t}$ = angular velocity = $\omega$;

$$\therefore\ v = \omega r \quad \dots\dots\dots\dots \quad (2)$$

In practice angular velocity is not expressed in radians per min. or per sec., but in revolutions per min. or per sec.

Now in one revolution the point moves through a distance $2\pi r$, so that if a point rotates uniformly at $n$ revolutions per sec., the velocity at a radius $r$ is given by

$$v = 2\pi r n \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

NUMERICAL EXAMPLE.—*If a shaft* 4 *in. in diameter rotates at a uniform rate of* 80 *revolutions per min., what is the peripheral velocity of the shaft in ft. per sec.?*

In this case $r = 2$ in., $n = \frac{80}{60}$ per sec.

$\therefore$ Peripheral velocity $v$ in in. per sec.

$$= 2\pi r n$$
$$= 2 \times 3{\cdot}1416 \times 2 \times \tfrac{80}{60}$$
$$= 16{\cdot}76.$$

$\therefore$ Peripheral velocity $= \dfrac{16{\cdot}76}{12}$

$$= 1{\cdot}40 \text{ ft. per sec.}$$

**Centripetal and Centrifugal Force.**—If a body moves with uniform velocity $v$ ft. per sec. in a circle of radius $r$ ft.,

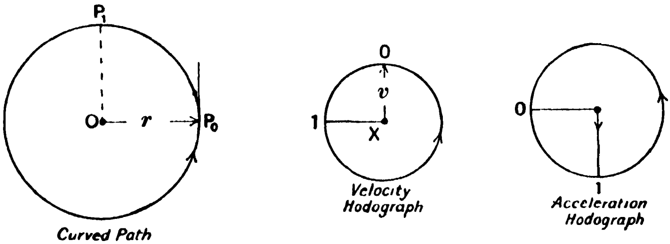


FIG. 53.—Centripetal Acceleration.

Fig. 53, the velocity hodograph will be a circle of radius $v$, the radius X 0 of the hodograph being at right angles to the corresponding radius O $P_0$ of the curved path. When the point in the curved path has reached $P_1$, the radius has turned through a right angle, and in reaching the corresponding point 1 on the velocity hodograph, the radius has turned through the same angle. The velocity hodograph,

therefore, turns through a complete circle in the same time as the body moving in the curved path completes its circle.

The acceleration hodograph will also be a circle, because it is obtained from the velocity hodograph by the same construction as that employed for drawing the former. The radius at 0 in the acceleration hodograph is parallel to the tangent at O and is thus at right angles to X 0 and opposite to the radius O $P_0$ of the curved path; a revolution of the acceleration hodograph will also be completed when one revolution in the curved path is completed.

We get, therefore, the result that with uniform motion in a circle there is a constant acceleration towards the centre. This acceleration is usually called the *centripetal acceleration*; its magnitude can be found as follows:

Let $t$ secs. be the time taken to complete the circle, then we have

$$v = \frac{2\pi r}{t}; \quad \dots\dots\dots\dots\dots\dots \quad (4)$$

also the acceleration $a$ is the velocity on the velocity hodograph;

$$\therefore\ a = \frac{2\pi v}{t} \quad \dots\dots\dots\dots\dots\dots \quad (5)$$

Dividing, we get

$$\frac{v}{a} = \frac{r}{v},$$

or

$$a = \frac{v^2}{r} \quad \dots\dots\dots\dots\dots\dots \quad (6)$$

If the weight of the body is W, we have, by the rule

$$\text{Force} = \frac{\text{weight} \times \text{acceleration}}{g},$$

a constant "centripetal force" acting towards the centre of the circle to maintain the motion.

The force equal and opposite to this, which is the apparent force acting outwards upon the body, is called the "centrifugal force," the two terms being often confused.

The centrifugal force is really the force acting outwards at the same radius as the rotating body which will equilibrate or balance the system of forces acting on the body, as the consideration of the equilibrium of such bodies is correctly dealt with by considering the forces acting on the body together with the reversed radial accelerating force as forming a system in equilibrium.

Since the weight of a body acts at the centre of gravity, and the centrifugal force acting on each portion of the body is proportional to the weight of that body, it follows that the resultant centrifugal force also acts through the centre of gravity.

We get, therefore, from equation (6),

$$\text{Centripetal or centrifugal force} = F_c = \frac{W a}{g}$$

$$= \frac{W v^2}{g r} \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

If the velocity is given in terms of revolutions per min. (N), we have, since

$$v = \frac{2\pi r N}{60},$$

$$F_c = \frac{4\pi^2 N^2 r^2 W}{3{,}600 g r}$$

$$= \frac{4\pi^2 N^2 r W}{3{,}600 g} \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

$$= 0{\cdot}00034 N^2 r W \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

NUMERICAL EXAMPLES.—(1) *What force acting horizontally tends to overturn a train weighing* 100 *tons when running round a curve of* 500 *ft. radius at* 60 *miles per hour?*

In this case

$$W = 100 \text{ tons},$$

$$v = 60 \text{ miles an hour}$$

$$= 88 \text{ ft. per sec.},$$

$$r = 500 \text{ ft.}$$

∴ Centrifugal force which tends to overturn the train

$$= F_c = \frac{W v^2}{g r}$$

$$= \frac{100 \times 88 \times 88}{32 \cdot 2 \times 500}$$

$$= \underline{48 \text{ tons nearly.}}$$

(2) *At how many revolutions per min. must a stone weig ing $\frac{1}{4}$ lb. whirl horizontally at the end of a string 5 ft. long cause a tension of 2 lb. in the string?*

In this case

$$W = \tfrac{1}{4},$$
$$F_c = 2,$$
$$r = 5.$$

∴ Using equation (9),

$$2 = 0 \cdot 00034 \, N^2 \, . \, 5 \, . \, \tfrac{1}{4},$$

$$N^2 = \frac{8}{5 \times 0 \cdot 00034},$$

$$N = \underline{68 \cdot 6 \text{ revolutions per min.}}$$

**Work Done on Rotating Bodies; Brake Horse power.**—We have considered the work done on bodi moving in various ways, bu have not dealt with the speci case of bodies rotating un formly, such as we have in th case of belt-driven pulleys.

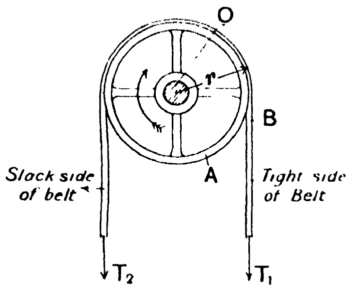


FIG. 54.

Referring to Fig. 54, suppos that there is a tension $T_1$ ll on the tight side of the bel and a tension $T_2$ on the slac side. If the pulley has a radiu of $r$ ft. to the centre of th belt and makes one revolution, and the belt does not sli the whole of the work done must be absorbed by th pulley.

The work done by the tension $T_1 = 2\pi r \, T_1$, and the wor

done against the tension $T_2 = 2\pi r T_2$; therefore the resulting work done on the pulley in one revolution

$$= E = 2\pi r T_1 - 2\pi r T_2$$
$$= 2\pi r (T_1 - T_2) \quad \ldots\ldots\ldots \quad (10)$$

Now suppose that the pulley makes N revolutions in 1 min.; then the work done on the pulley in 1 min.

$$= E \times N = 2\pi r N (T_1 - T_2) \quad \ldots\ldots\ldots \quad (11)$$

$$\therefore \text{ Horse-power transmitted} = \frac{\text{work per min.}}{33{,}000}$$
$$= \frac{2\pi r N (T_1 - T_2)}{33{,}000} \quad \ldots\ldots \quad (12)$$

The *brake horse-power* (written B.H.P.) of an engine is the horse-power given out by the engine as opposed to the indicated horse-power,* which is the horse-power put into the cylinder; it is so called because it is often measured by a friction brake applied to the fly-wheel of the engine. This friction brake in its most common form may be regarded as the converse of the belt drive.

If W is the weight on one end of the brake, and $w$ is that on the other, and $r$ is the radius of the fly-wheel measured to the centre of the rope or belt, we shall have, by exactly similar reasoning to that previously given,

$$\text{B.H.P.} = \frac{2\pi r N (W - w)}{33{,}000} \quad \ldots\ldots\ldots \quad (13)$$

**Work Done by a Couple upon a Rotating Body.**—We assume that the reader has already made himself aware of the fact that two equal and opposite parallel forces are said to form a "couple," and that the moment of a couple is called a "torque" or "twisting moment," and is measured by the product of the magnitude of one of the forces and the perpendicular distance between them. It can also be proved quite easily that a couple is equivalent to any other couple in the same plane having the same moment; we can, therefore,

* See p. 14.

imagine a couple of twisting moment $T_M$ to be made up of two equal and opposite forces F, one of which acts at a point A at radius $r$ on a rotating body, and the other at the centre O, about which the body is rotating, so that $T = F a$.

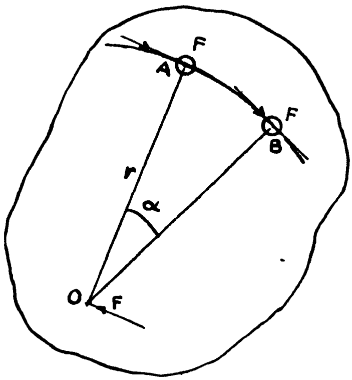


FIG. 55.

Suppose that after a very short time the point A has moved to B, Fig. 55.

The distance moved by the body under the action of the upper force $F = F \times$ arc $AB = F r a$.

The point O does not move, so that the lower force F does no work upon the body.

We therefore see that the work done on the body

$= F r a = T_M$.

This is the work done in moving through the angle $a$; if the motion is uniform and the angle turned through per sec. is $\omega$, we see that

Work done per sec. $= T_M \omega$ . . . . . . . . . . . . . . . . . . . . (14)

= twisting moment × angular velocity.

**Efficiency of a Turbine.**—We will now apply the principles which we have explained to determine the theoretical efficiency of a turbine. Fig. 56 shows diagrammatically one pair of vanes of a water turbine of the "outward-flow" type; the stream of water flows along the guide vane A B and impinges against the moving vane B C with a velocity $v$, driving it in a counter-clockwise direction, and is thrown out from the moving vane with a velocity V. If the water is to enter the moving vane without shock, the inclination of the vane to the tangent to the circle at B must be equal to the inclination of the relative velocity of the water to the moving vane.

*Velocity Diagrams at Inlet and Outlet.*—Let $v_t$ be the tan-

gential velocity at the point B. If now we set out $a b$ to represent the inlet velocity $v$ of the water, and set out $b c$ to represent a velocity equal and opposite to the tangential velocity $v_t$, it follows from our previous explanation that $a c$ will represent the velocity relatively to the moving vane, the inlet angle of which should be equal to $\phi$ to avoid shock at entry. To complete the diagram we draw $a d$ perpendicular to $b c$ produced. Then $b d$ represents the tangential com-

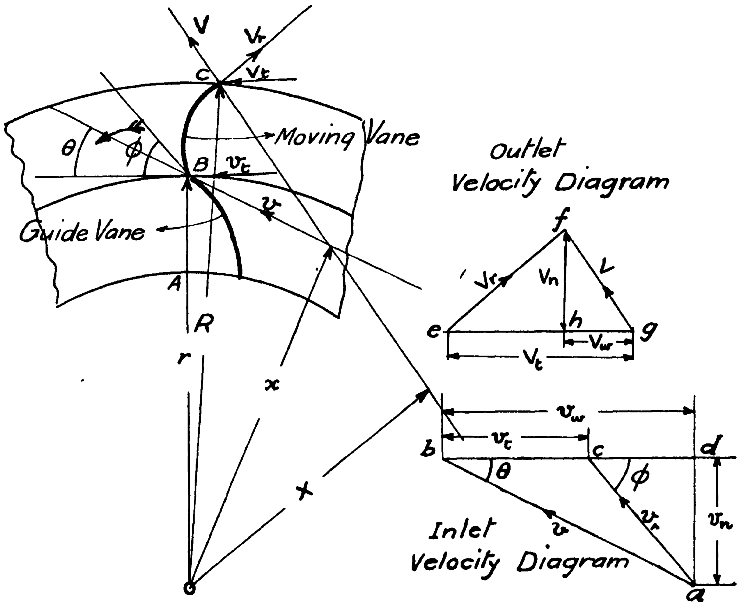


Fig. 56.

ponent of the inlet velocity and is called the *inlet velocity of whirl* ($v_w$). Coming to the outlet, we draw $g e$ to represent the tangential velocity $V_t$, and $e f$ to represent the velocity $V_r$ of the water relative to the vane; then $f g$ must represent a velocity equal and opposite to the actual velocity V with which the water leaves the vane. The tangential component $g h$ of this outlet velocity is called the *outlet velocity of whirl* ($V_w$).

*Efficiency of Turbine.*—To ascertain the efficiency of the turbine we must determine the work done per lb. of water flowing.

The kinetic energy of a weight at inlet is equal to $\frac{W v^2}{2g}$, and at outlet is $\frac{W V^2}{2g}$.

$$\therefore \text{ Work done per lb.} = \frac{v^2 - V^2}{2g} \quad \ldots\ldots\ldots\ldots\ldots (15)$$

If W lb. of water are supplied to the turbine from a height or "head" $h$, the work supplied is $W h$;

$$\therefore \text{ The work supplied per lb.} = h \quad \ldots\ldots\ldots\ldots\ldots (16)$$

$$\therefore \text{ Efficiency of turbine} = \frac{\text{work done}}{\text{work supplied}}$$

$$= \frac{v^2 - V^2}{2gh} \quad \ldots\ldots\ldots (17)$$

This problem may also be attacked from another point of view, as follows:

A pound of water per sec. impinges on the vane at B with velocity $v$, and a pound is thrown off with velocity V.

$\therefore$ Gain of velocity per sec. at B = acceleration = $v$,
and ,, ,, ,, ,, C = ,, = $-$ V.

Since by the Newtonian law force $= \frac{\text{weight} \times \text{acceleration}}{g}$, we have an equivalent force at B $= \frac{1 \times v}{g}$, and at C $= -\frac{1 \times V}{g}$.

Taking moments round O, we have a turning effect giving a twisting moment $= T_M = \frac{v x}{g} - \frac{V X}{g}$.

$$\left.\begin{array}{l}\text{Now } v x = v_w r \\ \text{and } \quad V X = V_w R\end{array}\right\} \text{ by the law of similar triangles.}$$

$$\therefore T_M = \frac{v_w r - V_w R}{g} \quad \ldots\ldots\ldots\ldots (18)$$

$\therefore$ Work done per lb.

$$\text{per sec.} = \text{twisting moment} \times \text{angular velocity}^{*}$$
$$= \mathrm{T_M}\,\omega$$
$$= \frac{\omega\,(v_w\,r - \mathrm{V}_w\,\mathrm{R})}{g}$$
$$= \frac{v_w\,.\,\omega\,r - \mathrm{V}_w\,.\,\omega\,\mathrm{R}}{g};$$

but $\qquad \omega\,r = \text{tangential velocity at B} = v_t,$

and $\qquad \omega\,\mathrm{R} = \quad ,, \quad ,, \quad ,, \ \mathrm{C} = \mathrm{V}_t.$

$$\therefore \text{ Work done per lb. per sec.} = \frac{v_w\,v_t - \mathrm{V}_w\,\mathrm{V}_t}{g} \quad \ldots\ldots \ (19)$$

$$\therefore \text{ Efficiency of turbine} = \frac{v_w\,v_t - \mathrm{V}_w\,\mathrm{V}_t}{g\,h} \quad \ldots\ldots \ (20)$$

* See p. 98.

# CHAPTER VI

## VELOCITIES AND ACCELERATIONS IN MECHANISMS

**Velocity Diagrams.**—We have considered up to the present the application of graphical methods to the consideration of the motion of bodies as a whole; we now come to the case in which we wish to compare the relative motions of different parts of the same body and also the relative

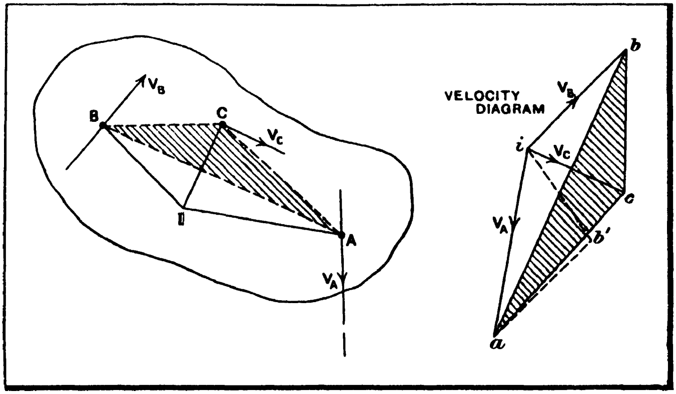


FIG. 57.—Instantaneous or Virtual Centre.

motions of different elements constituting what is known as a "mechanism."

**Instantaneous or Virtual Centre.**—Let Fig. 57 represent any body having motion in its plane, and let A and B be any two points in it, and at the moment at which the body has the position shown let the velocity of the point A

be $V_A$ in the direction indicated, and let $V_B$ be the velocity of the point B.

Choosing any pole $i$, and choosing a suitable velocity scale, set out $i\,a$ equal and parallel to $V_A$, and $i\,b$ equal and parallel to $V_B$, and join $a\,b$.

Now, since velocities are vector quantities, *i.e.* they can be represented in magnitude and direction by straight lines, they must follow the same law as forces, *i.e.* $i\,a$ represents the resultant of velocities $i\,b$ and $b\,a$; $b\,a$, therefore, represents the velocity that must be combined with $V_B$ to give $V_A$, or, in other words, $b\,a$ is the relative velocity of B to A. But B and A are fixed points upon a rigid body, and they can have no relative motion along their lines of junction A B; if they did, the length A B would get less or greater, which is contrary to the assumption that the body is rigid. The only relative motion which can take place between the points, therefore, is at right angles to A B, so that we see that $a\,b$ must be at right angles to A B.

In this connection we should note carefully that $a\,b$ is not the resultant of the velocities $V_A$, $V_B$. If we had wanted to find the resultant, we should have drawn $a\,b'$ equal to $V_A$ and joined $i\,b'$, as shown in dotted lines. Now, take any other point C on the body, and draw $i\,c$ to represent its velocity $V_C$; then, by exactly the same reasoning, since C A and C B are fixed lengths, we see that $c\,a$ must be perpendicular to C A and $c\,b$ perpendicular to C B.

The figures A B C and $a\,b\,c$ are exactly similar and turned through a right angle relatively to each other.

Now, consider a point I in the figure that is perpendicular to both $V_A$ and $V_B$. Then the figures I A B C and $i\,a\,b\,c$ are exactly similar, and I corresponds to $i$. But the velocity of any point in the figure is given to scale in the velocity diagram by the distance from the pole $i$ to the corresponding point in the velocity diagram; this distance for the point I is zero, so that the point I has no velocity. This point, therefore, corresponds to the point about which the body is rota-

ting at the instant, and is called the *virtual* or *instantaneous centre*.

If this instantaneous centre I is known, we get the following rule for drawing the velocity diagram for the whole body:

Redraw the body turned through 90° and changed in size, so that

$$\frac{ia}{\mathrm{IA}} = \mathrm{V_A},$$

*i.e.* the second figure is drawn to the velocity scale instead of the space scale.

The figure thus obtained gives, by measuring the distance of any point from $i$, the velocity of the corresponding point in the figure.

We note, as follows from the idea of rotation about the instantaneous centre, that the velocity of any point is at

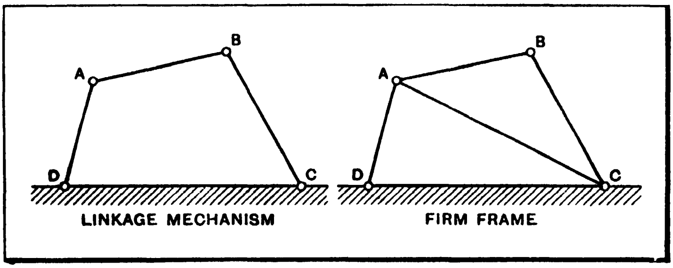


FIG. 58.—Linkage Mechanism and Firm Frame Structure.

right angles to the line joining the point to the instantaneous centre.

**Application to Linkage Mechanisms.**—A linkage mechanism consists of a number of bars pivoted together so as to transmit a definite motion. This collection of pin-jointed bars is called a "kinematic chain," and on fixing one of the links it becomes converted into a "mechanism." It is the dynamic equivalent of the framed structure of graphic statics, and the velocity diagram is the equivalent of the reciprocal figure used to determine the forces acting in a framed structure.

A linkage mechanism can be converted into a firm frame structure by the addition of one link or element without the addition of any pins or "nodes."

In the left-hand portion of Fig. 58 we show a kinematic chain which is known as the "four-bar chain," and which, by "inversion" or fixing of different links, gives a number of familiar mechanisms, such as the "beam-engine mechanism," the "Watt parallel motion," and the "drag-link coupling."

It has four members or links and four pins or nodes. By adding one link, such as A C, without adding to the number of pins, we get the firm frame shown in the right-hand portion of the figure.

The relation between the number $n$ of pins or nodes and

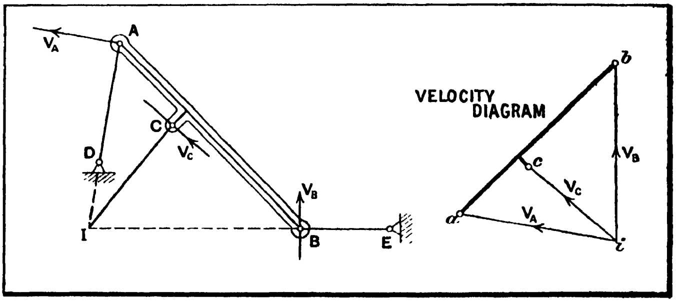


FIG. 59.—Linkage Mechanism and Velocity Diagram.

the number $b$ of links or elements for a linkage mechanism is given by the relation

$$b = 2n - 4.$$

In place of the actual links we often have in actual mechanism their kinematic equivalents; a straight slide, for instance, such as occurs in the ordinary crank and connecting-rod mechanism, is equivalent to a link of infinite radius.

Suppose that A B, Fig. 59, represents one link of a linkage mechanism. The end A is constrained by the link A D to move about the fixed point D, which is the pivot of the

link A D, and B is constrained to move about the fixed point E, which is the pivot of the link B E.

Then the velocity $V_A$ of A must be at right angles to A D, and the velocity $V_B$ of B must be at right angles to B E. Suppose that we know the velocity $V_A$, and wish to find $V_B$ and also the magnitude and direction of the velocity of a point C rigidly connected to the link A B.

Choosing a suitable velocity scale, set out *a i* on the velocity diagram to represent $V_A$. Then draw a line *i b* in the direction $V_B$, and draw *a b* perpendicular to the link A B; on this line *a b* draw the skeleton *a b c* of the link A B C as shown. Then *i c* gives the magnitude and direction of the required velocity of the point C.

It has not been necessary in this problem to find the instantaneous centre I of the link A B, but it may be determined as indicated in dotted lines by drawing A I, B I perpendicular to *a i* and *b i* respectively.

If only the direction of motion of C had been required, we could have found I at once by producing A D and B E; but we could not get the magnitude of its velocity without drawing the velocity diagram or making an equivalent graphical construction.

**Example of Quick-return Motion.**—We will further illustrate the treatment of velocity diagrams with reference to the crank and slotted-lever quick-return motion that is often employed in shaping machines. This mechanism is sometimes referred to as the "Whitworth quick-return motion"; it is not strictly this motion, however, but a modification of it.

Fig. 61 shows the mechanism in skeleton form, while Fig. 60 represents it in a form as actually employed in some heavy types of shaping machines in which the tool-carrying ram is movable transversely upon the machine frame; in other lighter types of shaping machines in which this mechanism is used the link K is usually either almost horizontal or inclined slightly upwards.

The feathered or splined shaft M drives a spur pinion F,

hich is carried by the carriage and gears with a similar othed wheel G. Projecting from this wheel is a pin C,

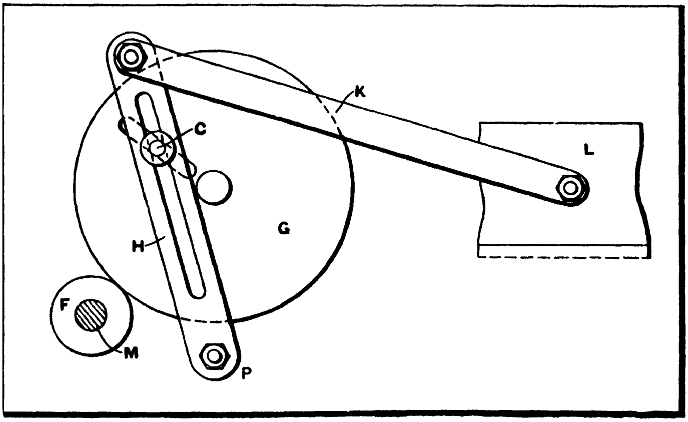


FIG. 60.—Quick-return Mechanism.

hich is radially adjustable to vary the stroke of the tool. is pin engages a slotted link H, which is pivoted by a pin

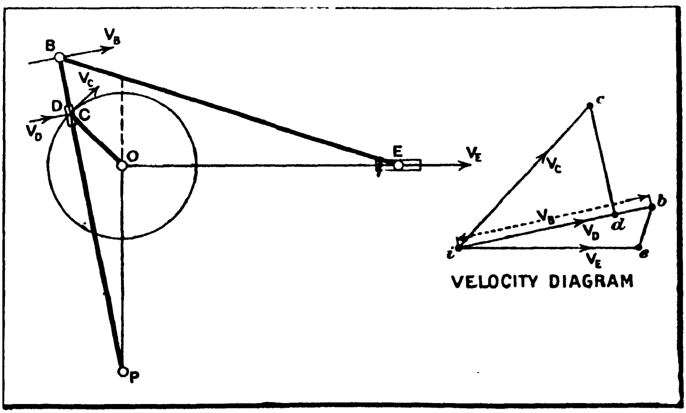


FIG. 61.—Velocity Diagram of Quick-return Motion.

to the carriage, and carries pivoted thereto at its upper end connecting-rod K, which is pivoted to the tool-carrying m L. The effect of this mechanism is to give a compara-

tively slow movement to the tool for its cutting stroke, and a comparatively quick movement for its return or idle movement.

In this case we know the velocity $V_C$ of the pin C; so to convenient scale set out *i c*, Fig. 61, to represent $V_C$ at right angles to the equivalent link O C. Calling D the point of contact of the pin in the slot, we know that D moves about the pin P, so draw *i d* perpendicular to P D; and we know also that the relative motion of C and D is along the slot, so draw *c d* parallel to the slot, thus fixing the point *d* on the velocity diagram.

Now produce *i d* to *b*, making

$$\frac{ib}{id} = \frac{PB}{PD}.$$

*i b* thus corresponds to the link P B drawn to the velocity scale at right angles to it.

$\therefore$ *i b* represents the velocity $V_B$ of the point B.

It will be noted that the pole *i* in this case corresponds to one of the points of the link.

Now draw *b e* perpendicular to B E, because we have seen that the relative velocity of two points upon the same link must be at right angles to the link. We also know that, since the tool-carrying ram is mounted in horizontal guides, $V_E$ must be horizontal. Therefore, draw *i e* horizontally, thus obtaining the point *e* on the velocity diagram.

The following dimensions have been taken in Fig. 61: P O = 9·6 in., P B = 14·88 in., and B E = 16·26 in., and the crank has been taken at 45° to the vertical. Taking $V_C = 0{\cdot}25$ ft. per sec., $V_E$, the tool speed, comes about $0{\cdot}95\ V_C = 0{\cdot}24$ ft. per sec.

**Variation of Cutting Speed.**—To obtain a complete study of a mechanism we require to know the manner in which the velocity of each of the parts varies during the complete cycle of the mechanism; this consideration of the variation of velocity is equivalent to the consideration of the variation of the loads or stresses in the members of a bridge

·uss for all possible positions of the load passing over the ridge. In the latter case we draw stress diagrams for each osition of the load, and show the result by tabulation or iagram, and we proceed in exactly the same manner in the resent case by drawing the velocity diagram for a number f positions of the rotating crank and plotting the velocities p at the corresponding points on the tool stroke.

Referring to Fig. 62, $E_i$, $E_o$ are the inner and outer posi- .ons of the tool slide, which correspond to positions of the

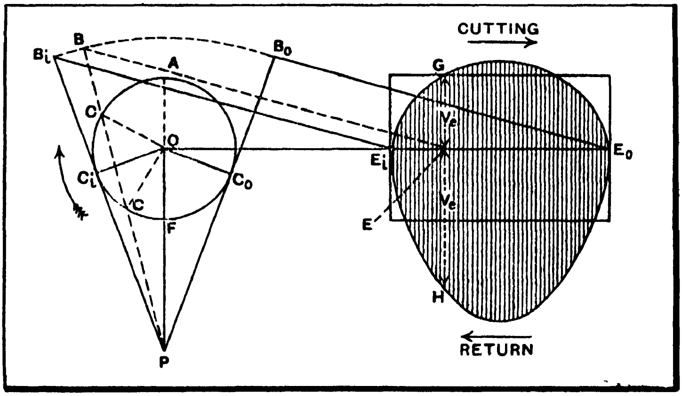


FIG. 62.—Diagram Illustrating Variation in Cutting Speed.

.otted lever B P, which are tangential to the crank circle 'C A. When, therefore, the crank-pin moves round the arc $_i$ A $C_o$, the tool performs its cutting stroke; and when the rank-pin moves round the arc $C_o$ F $C_i$, the tool performs its lle return stroke. Now the crank-pin moves with uniform elocity, so that the time taken for cutting and returning vill be proportional to the lengths of the corresponding arcs n the crank circle, *i.e.* to the half-angles subtended at the entre of the circle.

$$\frac{\text{Cutting time}}{\text{Return time}} = \frac{\text{angle } C_i O A}{\text{angle } C_i O F};$$

$$\frac{\text{Mean cutting speed}}{\text{Mean return speed}} = \frac{\text{angle } C_i O F}{\text{angle } C_i O A}.$$

In the present case this ratio curves equal to about 0·67.

Care should be taken to distinguish between the ratio of mean cutting and return speeds and that of the maximum cutting and return speeds.

In the present case the latter ratio comes about 0·47. The diagram of velocity variation shown in Fig. 62 is obtained as follows: Draw the oscillating slotted lever in any position P B, and find the corresponding point E on the tool stroke. There are two positions of the crank-pin C, indicated by the dotted lines O C, for this position of the tool, corresponding to the cutting and return strokes respectively. By means of the graphical construction previously described, find the cor-

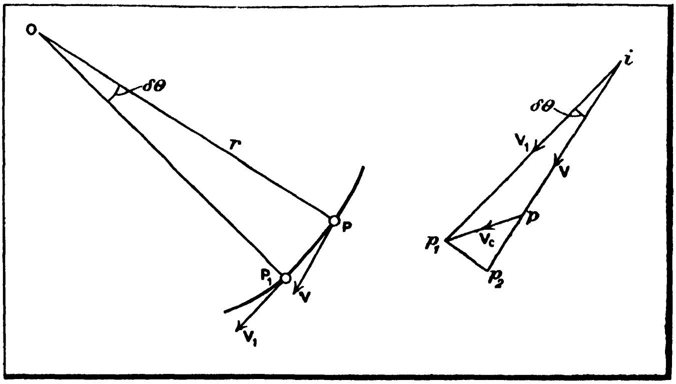


FIG. 63.—Diagram Illustrating Acceleration.

responding velocities $V_e$ of cutting and return and plot them up and down respectively upon the stroke base. Repeat this process for a number of points, and join up the points corresponding to G and H, thus getting the diagram of velocity variation as shown. We shall deal later with other examples of velocity diagrams, and will next pass on to the consideration of acceleration in mechanisms.

**Acceleration Diagrams.**—*General Nature of Acceleration.*—In addition to a knowledge of the velocities of various parts of a mechanism we need to know also the acceleration of the various portions.

Acceleration is defined as the rate of change of velocity; in the ordinary case of motion in a straight line the velocity changes in *magnitude* only, but, as we have already seen, when it moves in a curved path it changes in *direction* with or without changing in magnitude.

Suppose that at a certain instant a body is at a point P, Fig. 63, and is moving with a velocity V, and suppose that after a very short time $\delta t$ the body has reached $P_1$, and is moving with a velocity $V_1$. Then setting out $i p$ to represent V, and $i p_1$ to represent $V_1$, $p p_1$ represents the velocity which must be compounded with V to give $V_1$. Thus, $p p_1$ must be the change of velocity $V_c$.

The acceleration $a$, therefore, is equal to $\frac{V_c}{\delta t}$ and acts in the direction $p p_1$.

In the familiar case of uniform motion with velocity $v$ in a circle of radius $r$, the acceleration is due to change of direction only in the velocity, and acts towards the centre of the circle.

It is called the centripetal or radial acceleration ($A_r$), and is given by

$$A_r = \frac{V^2}{r}.$$

In the general case in which the velocity changes in magnitude as well as direction we may consider the acceleration as compounded of the direction and magnitude accelerations considered separately.

This is owing to the fact that accelerations are also vector bodies, so that the law of vector applies to them as well as to velocities.

Draw $p_1$, $p_2$ perpendicular to $i p$ produced; then, since the angle $\delta\theta$ is very small, $p_1$, $p_2$ may be considered as equivalent to an arc with centre $i$.

If O is the centre of curvature of the short piece $P P_1$ of the curved path, the direction or radial acceleration, *i.e.* that acting at right angles to V, will be given by $\frac{p_1 p_2}{\delta t}$, which will

be equal to $\frac{V^2}{r}$, the difference between $\frac{V^2}{r}$ and $\frac{V_1{}^2}{r}$ being negligible; the magnitude acceleration, *i.e.* the component of the acceleration acting in the direction of V, will be given by $\frac{p\,p_2}{\delta t}$.

**Accelerations of Points upon a Rigid Member or Link.**—Suppose that A and B are fixed points upon a rigid member or link, and suppose that the accelerations of $A_A$, $A_B$ of these two points are known in magnitude and direction.

Taking a pole $q$, Fig. 64, draw $q\,a_1$, $q\,b_1$ to represent these

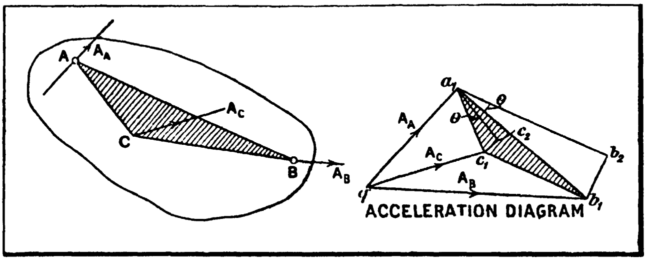


FIG. 64.—Acceleration Diagrams of Points upon a Rigid Member.

respective accelerations, and join $a_1$, $b_1$. Then $a_1\,b_1$ represents the relative acceleration of A and B.

But we have shown that the relative velocity of B to A must be at right angles to A B, *i.e.* it must be a rotation of B about A. The relative acceleration of B to A can be compounded into components $b_1\,b_2$ and $b_2\,a_1$ respectively at right angles and parallel to A B. Therefore, $a_1\,b_2$ represents the radial or centripetal acceleration of B along A B, *i.e.*

$$a_1\,b_2 = \frac{V_{AB}{}^2}{AB},$$

where $V_{AB}$ equals relative velocity of B to A ($b\,a$ in Fig. 57). The component $b_1\,b_2$ represents the acceleration in a direction at right angles to A B, this being what we have referred to as the magnitude acceleration. Now consider any other

point C upon the rigid member. Suppose that we know its acceleration $a\,c$, and that $q\,c_1$ is set out to represent it. Then $a_1\,c_1$ represents the relative acceleration of C to A: this can, as before, be divided into components $a_1\,c_2$ and $c_1\,c_2$. Since A C and B C are fixed in length, the only relative velocity possible is at right angles to them, *i.e.* a rotational velocity about the point A. Since B and C are fixed relatively, A C and A B must always turn through the same angle in equal times. Therefore the angular velocities and accelerations of B and C relatively to A must be equal.

We have not previously used the terms "angular velocity" and "angular acceleration," so we will just explain them more fully now.

If a body is rotating about a point, the angle turned through per sec. is called the angular velocity ($\omega$); the increase per sec. in this angular velocity is called the angular acceleration ($\omega_a$).

Then, since arc traversed per sec. = angle turned through per sec. × radius, we have

$$V = \omega\,r,$$
$$A = \omega_a\,r.$$

Now the radial or centripetal acceleration of C, *i.e.* $a_1\,c_2$,

$$= \frac{V_{AC}{}^2}{AC} = \frac{\omega^2\,AC^2}{AC} = \omega^2\,AC;$$

and the magnitude acceleration of C, *i.e.* $c_1\,c_2$,

$$= \omega_a \times AC.$$

$$\therefore \tan \angle\ c_1\,a_1\,c_2 = \frac{C_1\,C_2}{A_1\,C_2} = \frac{\omega_a}{\omega^2}.$$

Similarly, $$a_1\,b_2 = \omega^2\,AB,$$
and $$b_1\,b_2 = \omega_a\,AB;$$

$$\therefore \tan \angle\ b_1\,a_1\,b_2 = \frac{b_1\,b_2}{a_1\,b_2} = \frac{\omega_a}{\omega^2};$$

$$\therefore \angle\ c_1\,a_1\,c_2 = \angle\ b_1\,a_1\,b_2 = \theta.$$

Moreover, $$\frac{c_1\,a_1}{a_1\,b_1} = \frac{a_1\,c_2}{a_1\,b_2} = \frac{AC}{AB}.$$

Similarly, we can prove that $\frac{b_1 c_1}{a_1 b_1} = \frac{\text{B C}}{\text{A B}}$.

Summing up the result of our argument so far, we see th*the acceleration figure* $a_1 b_1 c_1$ *is similar to the origin figure* A B C, *and is turned through a constant angle* $\theta$ *who tangent is equal to* $\frac{\omega_a}{\omega^2}$, *where* $\omega_a$ *and* $\omega^2$ *are the angular acceler tion and velocity respectively of any point in the figure.*

In the above treatment we have assumed that two at lea of the accelerations are known ; we are now in a position investigate the manner in which acceleration diagrams f complete mechanisms can be drawn,

**The Crank and Connecting-rod Mechanism.**—Th mechanism is by far the most common of the link mechanisr in practical use, and will afford us a useful illustration of t application of the principles with regard to velocities a accelerations that we have been considering.

When we have investigated this mechanism, we shall fi that some of the results can be found by simple graphic constructions with which we may be already familiar, a which do not appear at first sight to involve the gene principles that are now under consideration. It is in t direction that the greatest progress has been made in rece years in the study of graphical methods of dealing wi mechanical problems; many of the graphical constructio have been known for many years, but they have been us as isolated constructions, apart from any reasoning bas upon graphical considerations, in a manner similar to t use of the empirical formulæ so common in practice.

The crank and connecting-rod mechanism, also known the slider-crank mechanism, comprises as essential featu a crank O C, Fig. 65, pivoted at C to a connecting-rod C the end B of which is guided in a straight line which usua passes through the axis O, about which the crank rotates.

Suppose that the crank-pin has a velocity $V_c$ when t mechanism is in the position shown in the figure. Th

choosing some convenient scale of velocity, we may draw $i c$ perpendicular to O C to represent $V_C$, the velocity of the crank-pin. By the construction that we have already explained we then draw $c b$ perpendicular to C B and draw $i b$ horizontal, *i.e.* parallel to the direction of movement of the point B. Then $i b$, measured on the velocity scale, gives us the actual velocity $V_B$ of B, usually called the piston velocity, and $c b$ gives the velocity of B relative to C.

Produce B C to meet the line through O perpendicular to B O in D, and consider the triangles $i b c$ and O C D. O D is

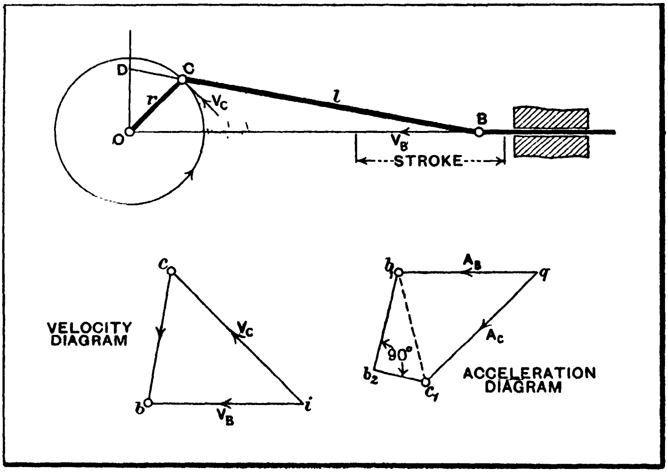


FIG. 65.—Crank and Connecting-rod Mechanism.

perpendicular to $i b$, O C to $i c$, and C D to $c b$. Therefore, these triangles are similar.

$$\therefore \frac{\mathrm{OD}}{\mathrm{OC}} = \frac{V_B}{V_C};$$

$$\therefore V_B = \frac{\mathrm{OD} \cdot V_C}{\mathrm{OC}} = \frac{\mathrm{OD} \cdot V_C}{r}.$$

If the crank-pin is rotating uniformly at N revolutions per sec.,

$$V_C = 2\pi N r;$$

$$\therefore V_B = 2\pi N \cdot \mathrm{OD}, \ldots\ldots\ldots\ldots (1)$$

so that O D represents the velocity of B, and this forms a very simple construction for finding it.

By similar reasoning, the velocity of B relative to C is given by

$$V_{BC} = cb = 2\pi N \, . \, CD \; \ldots\ldots\ldots\ldots \; (2)$$

**Acceleration Diagram.**—Next let us consider the acceleration of the point B, *i.e.* the piston acceleration. The point C is moving with a velocity $V_C$ in a circular path of radius $r$; if, as is usual, this velocity is constant in magnitude, its acceleration will be a radial or centripetal one only, and we have

$$\text{Acceleration of C} = A_C = \frac{V_C{}^2}{r} \; \ldots\ldots\ldots \; (3)$$

Choosing, therefore, a convenient scale of acceleration, set out $qc_1$ parallel to O C to represent $A_C$. Since the point B is guided so that it must move horizontally, its acceleration must be horizontal, so draw $qb_1$ in a horizontal direction. To fix the point $b_1$ we must consider the acceleration of the point B relatively to C, because we showed previously with reference to Fig. 64 that the acceleration of a point in a link is obtained by finding the vector sum of the known acceleration of some other point in the link and the relative acceleration of the two links. The relative acceleration of B to C is compounded of the relative radial or centripetal and magnitude accelerations, which act respectively along and at right angles to B C. We have shown from the velocity diagram that $cb$ is the relative velocity of B to C, and since the points are at a fixed distance apart, this velocity must be at right angles to B C. At the instant under consideration, therefore, the point B may be considered as rotating about the point C with a velocity equal to $cb$.

$$\therefore \text{ Radial acceleration of B relative to C} = \frac{cb^2}{BC} = \frac{cb^2}{l} \; \ldots\ldots \; (4)$$

Therefore, set out $c_1 b_2$ equal to $\frac{cb^2}{BC}$ and parallel to B C, thus fixing the point $b_2$. The magnitude acceleration is at

right angles to B C, and therefore to $c_1 b_2$, so that by drawing $b_2 b_1$ perpendicular to $c_1 b_2$ to intersect $q b_1$ in $b_1$, the latter point becomes fixed, and the length of $q b_1$ measured upon the acceleration scale will give us the required acceleration $A_B$ of the point B. In this way, by drawing the mechanism in various positions, we can obtain the velocity and acceleration of the piston for any point in its stroke. This method of drawing velocity and acceleration diagrams for mechanisms

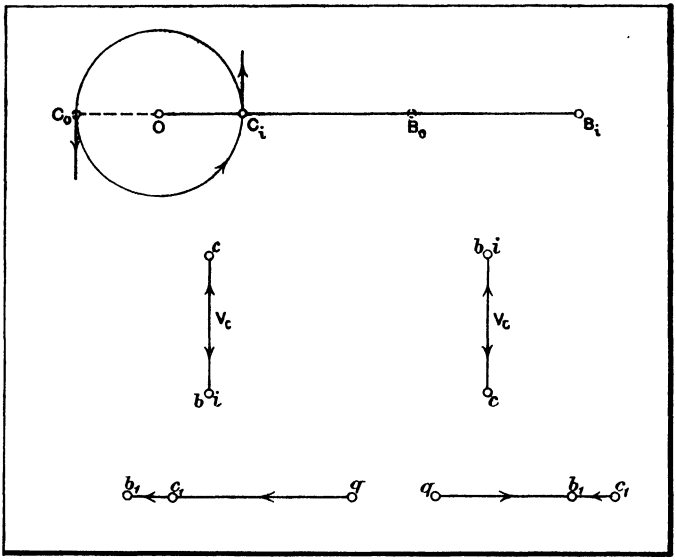


FIG. 66.—Acceleration in "In" and "Out" Positions.

was published by Prof. R. H. Smith, who called them "velocity and acceleration images"; it was applied to a number of mechanisms by Prof. S. Dunkerley in his book *Mechanism* (Longmans).

**Acceleration at Ends of Stroke.**—As we shall see later, the maximum acceleration of the piston occurs at the ends of its stroke. In this case the diagrams become so simple that we can obtain easily from them a formula for the accelera-

tions. Referring to Fig. 66, $B_i$ and $B_o$ represent the "in" and "out" positions of the point B, $C_i$ and $C_o$ being the corresponding positions of the crank-pin. In this case the velocity diagrams for each position reduce to straight lines $ic$, the points $b$ and $i$ coinciding. This makes the relative velocity of B to C vertical and equal to $V_C$. For the "in" position we have, as before, a radial acceleration of C equal to $\frac{V_C^2}{r}$, so that we set out $qc_1$ horizontally to represent this to a convenient scale. The radial acceleration of B relative to C will, as before, be parallel to BC and equal to $\frac{cb^2}{l}$, and will act towards the point C, and there can be no relative acceleration at right angles to BC, so that if we set out $c_1 b_1$ to represent $\frac{cb^2}{l}$, we shall get the point $b_1$, and $qb_1$ will give the acceleration of B.

For the "out" position we get a similar acceleration diagram as shown. We thus get

$$\text{Acceleration at "in" position} = qb_1$$
$$= A_i = qc_1 + b_1 c_1$$
$$= \frac{V_C^2}{r} + \frac{V_C^2}{l}$$
$$= V_C^2\left(\frac{1}{r} + \frac{1}{l}\right) \quad \ldots\ldots\ldots\ldots, (5)$$

$$\text{Acceleration at "out" position} = qb_1$$
$$= A_o = qc_1 - c_1 b_1$$
$$= \frac{V_C^2}{r} - \frac{V_C^2}{l}$$
$$= V_C^2\left(\frac{1}{r} - \frac{1}{l}\right) \quad \ldots\ldots\ldots\ldots (6)$$

If the crank-shaft makes N revolutions per sec. and $r$ is in feet, the velocity $V_C$ will be equal to $2\pi N r$ ft. per sec., so that $V_C^2 = 4\pi^2 N^2 r^2$.

So that equation (3) becomes

$$A_i = 4\pi^2 N^2 r^2 \left(\frac{1}{r} + \frac{1}{l}\right)$$

$$= \frac{4\pi^2 N^2 r^2}{r}\left(1 + \frac{r}{l}\right)$$

$$= 4\pi^2 N^2 r \left(1 + \frac{r}{l}\right).$$

The term $\frac{r}{l}$ is commonly called $n$, *i.e.* the connecting-rod is $n$ times as long as the crank.

We thus have

$$A_i = 4\pi^2 N^2 r \left(1 + \frac{1}{n}\right) \quad \ldots\ldots\ldots\ldots (7)$$

Similarly,

$$A_o = 4\pi^2 N^2 r \left(1 - \frac{1}{n}\right) \quad \ldots\ldots\ldots\ldots (8)$$

We will consider later the variation of the velocity and acceleration at the various points of the stroke, the curves representing these being those that we have already discussed in general terms as the "curve of action" and the "curve of resultant effort." We will now consider some of the better-known graphical constructions for finding the acceleration of the piston, and will show that they are in reality based upon the acceleration diagram shown in Fig. 65.

**Rittershaus Construction.**—This construction was first given in the French publication *Civilingénieur*, vol. xxv.

Referring to the upper diagram of Fig. 67, produce the connecting-rod B C to meet the perpendicular to O B in D; draw D G perpendicular to B D to meet B O produced in G and join C G; draw D H parallel to C G to meet B O produced in H; then the acceleration $A_B$ of the point B is given by $(GH - GO) 4\pi^2 N^2$, the lengths being measured on the space scale. We will prove this later.

**Elliott or Mohr Construction.**—This construction was given by Prof. Elliott in an interesting article in *Engineering*, vol. lxiii., p. 665, and was independently given by Prof.

Mohr, the well-known German authority on graphics. It is as follows:

Draw D P horizontal to meet O C produced in P; draw P N vertical to meet B C in N, and draw N J perpendicular to C B to meet B O in J.

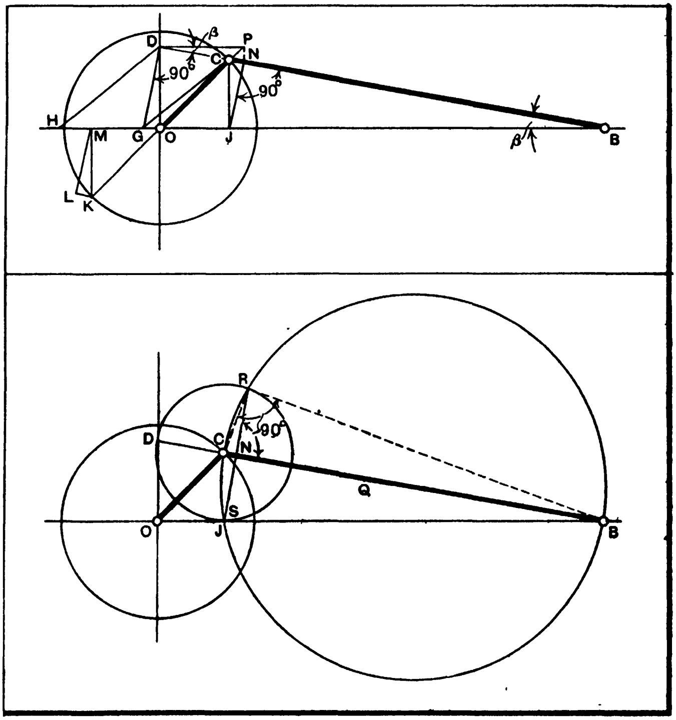


FIG. 67.—Rittershaus and Elliott or Mohr Construction (*upper diagram*); Klein or Kisch Construction (*lower diagram*).

Then $$A_B = 4\pi^2 N^2 O J.$$

To prove this construction we note that the triangles D C P and C O B are similar.

$$\therefore \frac{CD}{CB} = \frac{CP}{CO} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

The triangles C P N and C O D are also similar.

$$\therefore \frac{\mathrm{CP}}{\mathrm{CO}} = \frac{\mathrm{CN}}{\mathrm{CD}} \quad \ldots\ldots\ldots\ldots\ldots (10)$$

$\therefore$ [from (9) and (10)]

$$\frac{\mathrm{CN}}{\mathrm{CD}} = \frac{\mathrm{CD}}{\mathrm{CB}},$$

*i.e.*

$$\mathrm{CN} = \frac{\mathrm{CD}^2}{\mathrm{CB}} = \frac{\mathrm{CD}^2}{l} = \frac{k^2}{l}; \quad \ldots\ldots\ldots\ldots (11)$$

but from equation (2) $\quad V_{\mathrm{BC}} = 2\pi \mathrm{N} \,.\, \mathrm{CD}$;

$$\therefore \mathrm{CN} = \frac{V_{\mathrm{BC}}{}^2}{4\pi^2 \mathrm{N}^2 l} \quad \ldots\ldots\ldots\ldots (12)$$

Now refer back to the acceleration diagram of Fig. 65 and produce C O to meet the circle in K; draw K M parallel to $c_1 b_1$ and K L, L M parallel to $c_1 b_2$, $b_2 b_1$; then O K L M is a reproduction of $q c_1 b_2 b_1$ to a scale of reduction equal to

$$\frac{\mathrm{OK}}{\mathrm{A_C}} = \frac{r}{4\pi^2 \mathrm{N}^2 r} = \frac{1}{4\pi^2 \mathrm{N}^2}.$$

$$\therefore \mathrm{LK} = \frac{c_1 b_2}{4\pi^2 \mathrm{N}^2} = \frac{V_{\mathrm{BC}}{}^2}{4\pi^2 \mathrm{N}^2 l}$$

$\therefore$ [from (12)] $\quad \mathrm{CN} = \mathrm{LK}$.

Therefore, since the sides of the figures O C N J and O K L M are all parallel to each other, and two sides are equal, the remaining sides must be equal.

$$\therefore \mathrm{OJ} = \mathrm{MO} = \frac{\mathrm{A_B}}{4\pi^2 \mathrm{N}^2},$$

or

$$\mathrm{A_B} = 4\pi^2 \mathrm{N}^2 \,.\, \mathrm{OJ}.$$

To prove the Rittershaus construction we shall show that $\mathrm{OJ} = \mathrm{HG} - \mathrm{GO}$, and then the previous proof will hold. It will be noted that if $\mathrm{OJ} = \mathrm{HG} - \mathrm{GO}$, $\mathrm{GJ} = \mathrm{HG}$. To simplify our reasoning, let $\mathrm{CD} = k$ and $\angle\, \mathrm{DBO} = \beta$.

The triangles B G D, B J C are similar.

$$\therefore \frac{\mathrm{GJ}}{\mathrm{GB}} = \frac{\mathrm{DN}}{\mathrm{DB}};$$

$$\therefore \mathrm{GJ} = \frac{\mathrm{GB}\,.\,\mathrm{DN}}{\mathrm{DB}} = \frac{\mathrm{DN}}{\cos\beta} = \frac{k + \mathrm{CN}}{\cos\beta};$$

but $CN = \frac{k^2}{l}$ [from (11)],

$$\therefore GJ = \frac{k + \frac{k^2}{l}}{\cos\beta}$$

$$= \frac{kl + k^2}{l\cos\beta} = \frac{k}{l}\frac{(l+k)}{\cos\beta} \quad \ldots\ldots\ldots\ldots \quad (13)$$

Also, since CG and HD are parallel, the triangles HDB, GCB are similar.

$$\therefore \frac{GH}{GB} = \frac{CD}{CB} = \frac{k}{l}$$

$$\therefore GH = \frac{k}{l}\,.\,GB \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

Also $$\cos\beta = \frac{BD}{GB} = \frac{k+l}{GB};$$

$$\therefore GB = \frac{(k+l)}{\cos\beta};$$

$$\therefore GH = \frac{k}{l}\frac{(k+l)}{\cos\beta} \quad \ldots\ldots\ldots\ldots\ldots \quad (15)$$

$$= GJ \text{ [from (13)]}.$$

**Klein or Kisch Construction.**—This construction was given by Prof. J. F. Klein in a paper in the *Journal of the Franklin Institute*, U.S.A., 1891. It was also given by Kisch in 1890. Upon the connecting-rod as diameter describe a circle CRBS, and with C as centre and CD as radius describe a circle DRS, intersecting the previous circle in R and S. Join RS and let it intersect OB in J; then $A_B = 4\pi^2 N^2 OJ$. To prove this construction join CR, RB as indicated in dotted lines; then, since the angle CRD is a right angle owing to its being the angle in a semicircle, the triangles CRN, RBN are similar. Also CR = CD, because they are radii of the same circle.

$$\therefore \frac{CN}{CR} = \frac{CR}{CB};$$

$$\therefore CN = \frac{CR^2}{CB} = \frac{CD^2}{CB} = \frac{k^2}{l}.$$

But this is the same result as we obtained for the Elliott construction [equation (11)], for which we proved that

$$O J = \frac{A_B}{4\pi^2 N^2}.$$

Therefore, in the present construction also,

$$A_B = 4\pi^2 N^2 O J.$$

**Curves of Variation of Velocity and Acceleration.**—By drawing the velocity and acceleration diagrams for a number of positions of the crank and piston, we can draw curves showing the variation of these quantities for various positions of the mechanism. These curves can be plotted either as polar curves or upon a stroke base.

In Fig. 68 we show the polar velocity curve, from which it is clear that it consists of two loops. For any given angular position of the crank, say C, we join O C, cutting the loop in E; then $V_B = 2\pi N \cdot O E$, O E being measured upon the space scale. Similarly for the piston $C_1$, $V_B = 2\pi N O E_1$. If the connecting-rod were infinitely long, the point B would receive a simple harmonic motion, and the two loops would become circles touching the crank circle at the top and bottom and also the horizontal diameter.

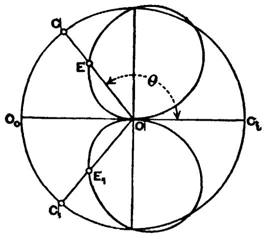

FIG. 68.—Polar Velocity Diagram.

In Fig. 69 the velocity and acceleration curves are drawn upon a stroke base; for any position B along the stroke the velocity $= V_B$, and the acceleration $= A_B$.

We have previously described a curve of velocities plotted upon a base of distances or displacements as a curve of action, and the similar curve for accelerations a curve of acceleration or resultant effort, and have proved that the subnormal to the curve of action gives the acceleration. As a rough check upon our working, therefore, draw a tangent to the velocity curve at the point F and draw F G at right

angles to it; then BG, the subnormal, should give the acceleration to a scale 1 in. $= \frac{z^2}{y}$, where 1 in. $= z$ ft. per sec. is the velocity scale, and 1 in. $= y$ ft. the space scale.

To obtain the resultant effort, generally called in this case

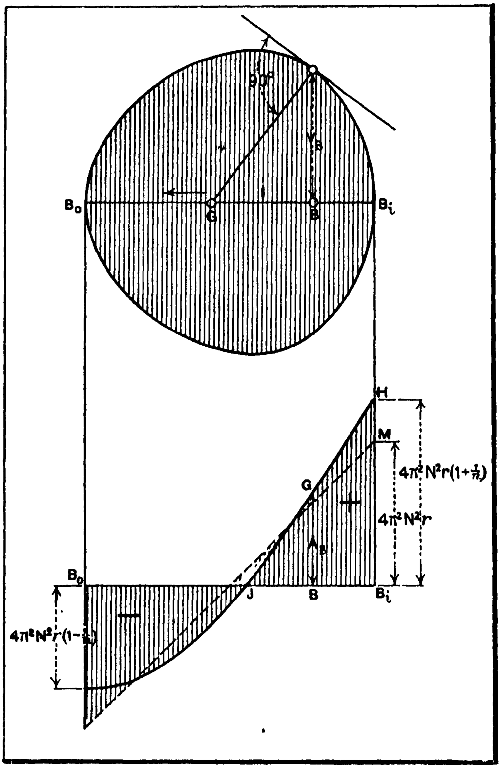


FIG. 69.—Velocity and Acceleration Diagrams for Piston.

the *inertia force*, we multiply the acceleration in feet per sec. per sec. by $\frac{W}{32 \cdot 2}$, where W is the weight of the reciprocating parts. We have also shown that the area of the curve of resultant effort between two points gives us the resultant work done upon the moving body; this work is used in

changing the kinetic energy of the body. In the present case the velocity of the reciprocating parts is zero at $B_o$ and $B_i$, so that there is no change in the kinetic energy over the whole stroke. This means that the resultant work done is zero, so that the total area of the acceleration curve must be zero, *i.e.* the areas marked $+$ and $-$ must be equal; this gives an important check upon our constructions. In the important application of these constructions to the dynamics of the steam- and gas-engine it is common to calculate the acceleration accurately at the two ends of the stroke and to draw in the actual curve approximately; so long as care is taken to preserve the rule as to the equality of area of the two parts of the curve, this will not result in serious error.

If the motion of the point B were a simple harmonic one, as would occur with a connecting-rod infinitely long, we should have, as previously explained, an ellipse for the curve of action, and a straight line for the curve of acceleration or resultant effort; this straight line would come as shown in dotted lines in Fig. 70, the acceleration at each end being $4\pi^2 N^2 r$.

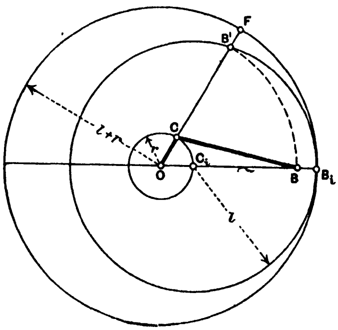


Fig. 70.—Zeuner's Construction for Displacement.

**Zeuner's Construction for the Point B.**—The following graphical construction due to Zeuner is very useful in problems on the crank and connecting-rod mechanism, because it saves constant striking off with the compasses from a given crank-pin position to obtain the corresponding position of the slide:

First describe the crank circle with centre O, Fig. 70, and radius $r$; then describe a circle with centre O and radius O $B_i$, *i.e.* $l+r$; and finally, with centre $C_i$, draw a circle with radius $C_i B_i$, *i.e.* $l$. Take any position of the

crank-pin, say C, and join O C and produce it, meeting the other circles in B′ and F respectively; then F B′ is the displacement of the point B from its "in" position $B_i$ for the given position C of the crank-pin. If the actual position of B is required, it can be swung down by means of an arc, as shown; but it is usually preferable to set out $B_i B$ equal to F B′ by means of a pair of dividers.

**Velocity and Acceleration of any Point in the Connecting-rod.**—In problems upon the forces acting upon

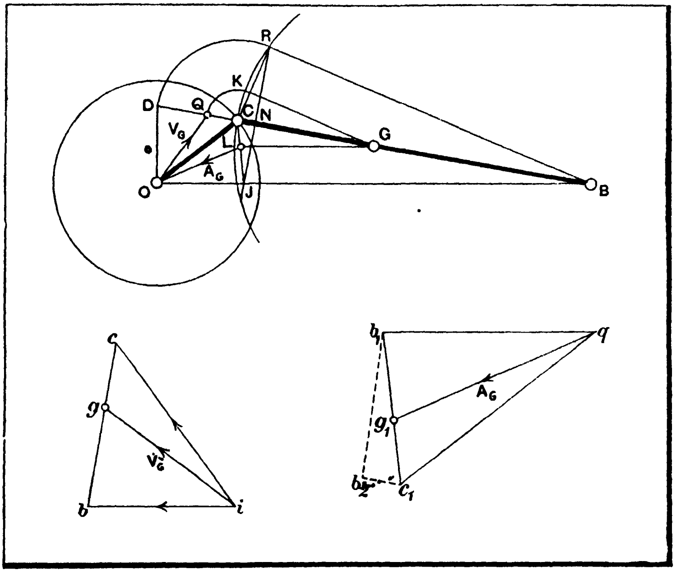


FIG. 71.—General Velocity and Acceleration of Connecting-rod.

the connecting-rod and in the analysis of valve-gears, in which motion is taken from a point upon the connecting-rod, we require to know the velocity and accelerations of such a point. This is very easily done by means of the relations which we have already shown to exist between a rigid body and its velocity and acceleration diagrams.

Referring to Fig. 71, suppose that we require to find the velocity and acceleration of the point G; then, if *i b c* and

$q c_1 b_1$ are the velocity and acceleration diagrams for the mechanism obtained as described with reference to Fig. 65, $b c$ and $b_1 c_1$ are what Prof. R. H. Smith called the velocity and acceleration images of the connecting-rod. If we divide $b c$, $b_1 c_1$ at points $g g_1$, so that $\frac{c g}{c b} = \frac{c_1 g_1}{c_1 b_1} = \frac{\mathrm{CG}}{\mathrm{CB}}$, we shall have the points corresponding to the point G, whose velocity and acceleration are required. If, therefore, we join $i g$ and $q g_1$, we shall have velocity $V_G$ and acceleration $A_G$ required. We can apply this principle as follows to the case in which one of the other constructions, say Klein's, is used for the acceleration diagram OCJ, and BC is produced to D to obtain the velocity diagram OCD: Through G draw GL parallel to BO to meet CJ in L, and join OL; then LO is a measure of the acceleration G, *i.e.* $A = 4\pi^2 N^2 \mathrm{LO}$. To get $V_G$ join BR, and draw GK parallel to it, intersecting CR in K. With centre C swing down CK on to CD, thus obtaining the point Q, and join OQ; then the velocity $V_G$ of G will be equal to $2N \cdot \mathrm{OQ}$ and will act at right angles to OQ. The next point that requires consideration is the distinction between the resultant effort or inertia force, as it is often called, and the static force transmitted by the mechanism.

# CHAPTER VII

## STATIC FORCES IN MECHANISMS

We have seen how the velocities and accelerations of any point in one of the members in a mechanism may be found, and from our previous considerations we have realised that if we know the acceleration of a body, we can find the resultant effort upon it by multiplying the acceleration by $\frac{W}{g}$, where W is the weight of the body and $g$ the gravity acceleration (32·2 in the pound-foot-second units). But in mechanism we require to know the gross effort upon any member so that we can find the resistance. Now in a machine we are really concerned more with the gross effort and the resistance than with the resultant effort (or *inertia force*, as it is very often called). We will consider these forces in detail.

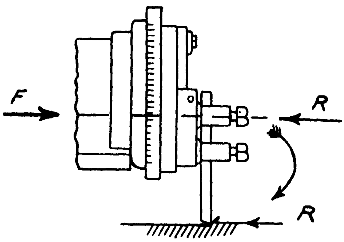

FIG. 72.

Suppose that A, Fig. 72, represents the moving part of a machine, say the ram of a shaping machine. Then if F is the gross effort transmitted by the machine to the ram, and R is the resistance to the tool offered by the work, F acts down the centre line of the ram, and R acts along the tool edge and is transferred up to the line of action of R by the

bending resistance of the tool, thus introducing a "couple," which we have indicated by the curved arrow, and which represents the bending moment in the tool.

Now $(F - R)$, which is the resultant effort upon the ram, must, by Newton's law of motion, be equal to the $\frac{Wa}{g}$, where $a$ is the acceleration of the ram, and can be found, when required, by one of the graphical constructions which we have previously given. In practice we shall require to know either the gross effort required to overcome a given resistance or else the resistance which a given gross effort can overcome; and if the resultant effort or inertia force $\frac{Wa}{g}$ is known, either of the other quantities can be found if one of them is given. It will be noted that when the machine is stationary the acceleration $a$ is zero, so that $\frac{Wa}{g} = 0$, and therefore $F = R$, so that F can be found by considering the balance of the forces in the mechanism when it is stationary. This is called a consideration of the static forces in the mechanism, and we require a method of finding the force in one link statically equivalent to a force in another.

**The Method of Virtual Velocities.**—One method of finding the static forces, when the velocities of various points are known, is to consider the work done in an imaginary very small displacement. If no work is spent in friction in the machine or in changing the kinetic energy of the parts, the work done by the effort applied at one point in the mechanism must be equal to the work done against the resistance at the other.

Let $F_A$, Fig. 73, be the effort at any point A of a mechanism which has to be balanced statically by a resistance $R_B$ at a point B, and which is, therefore, equivalent to an effort $F_B$ at B; and suppose that the velocities $V_A$ and $V_B$ of A and B are known in magnitude and direction. Then, in a very short time $t$, we have

$$\text{Work done by } F_A = \begin{cases}\text{component of } F_A \text{ in direction of V} \\ \quad \text{multiplied by } V_A \,.\, t\end{cases}$$
$$= Ac \times V_A \times t.$$
$$\text{Work done by } R_B = \begin{cases}\text{component of } R_B \text{ in direction of V} \\ \quad \text{multiplied by } V_B \,.\, t\end{cases}$$
$$= Bd \times V_B \times t.$$

If these are equal, we have

$$Ac \times V_A = Bd \times V_B;$$
$$\therefore\ Bd = \frac{Ac \times V_A}{V_B}.$$

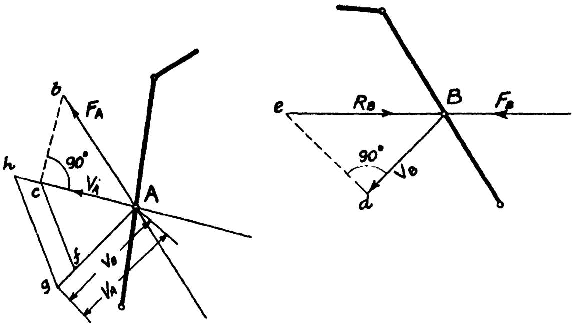


FIG. 73.

This can be found graphically by setting out $Ag$ in an convenient direction to represent $V_A$, and $Af$ to represen $V_B$ to the same scale; then join $fc$ and draw $gh$ parall to it to meet $Ac$, produced if necessary, in $h$.

Then
$$\frac{Ah}{Ac} = \frac{Ag}{Af} = \frac{V_A}{V_B},$$

*i.e.*
$$Ah = \frac{Ac \times V_A}{V_B};$$
$$\therefore\ Ah = Bd.$$

Therefore, by drawing $Bd$ to a convenient force scale, an drawing $de$, at right angles to it, to meet the line of actio of $R_B$ in $e$, we have $eB = R_B$.

**Example of Quick-return Motion.**—Take, for in stance, the shaping machine quick-return motion for whic

the velocities were found in Figs. 61 and 62. We have shown that $V_E$, Fig. 74, $= 0{\cdot}95\,V_C$. Suppose that the equivalent effort $F_E$ to be applied to the tool is 1,200 lb., and we require to find the force $F_C$ to be exerted at the crank-pin C. In this case $F_C$ and $F_E$ are both in the direction of their velocities, so that we have

$$F_C = \frac{V_E F_E}{V_C} = \frac{1{,}200 \times 0{\cdot}95}{1{\cdot}00} = \underline{1{,}140 \text{ lb. approx.}}$$

**Method of Instantaneous Centres.**—In cases in which the velocities of the various parts have not been determined the following method of finding the static forces can be employed; but before explaining the method we will deal

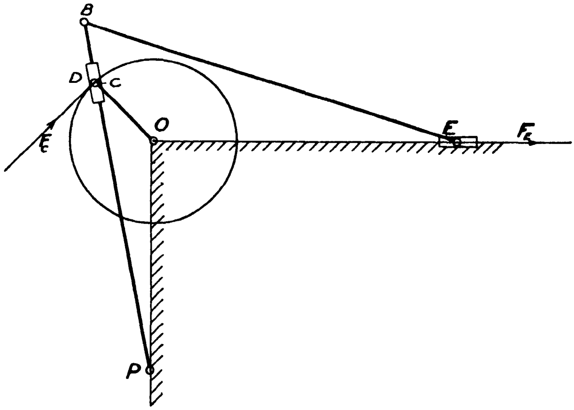


FIG. 74.

further with the manner in which the instantaneous centres can be found.

The instantaneous or virtual centre of any link or member *a* of a mechanism with relation to any link *b* is, as we have seen already,* the point about which the first link *a* may be considered as rotating at any instant if the second link *b* is considered fixed; it is written $I_{ab}$, and may be regarded as a point in either link. In the case of two links pivoted

* P. 102.

together the instantaneous centre is clearly at their pivot; instantaneous centres of this kind are often called "fixed centres."

*Three Virtual Centres in a Straight Line.*—If we have three links *a*, *b*, *c*, they will have three virtual centres, viz. $I_{ab}$, $I_{ac}$, and $I_{bc}$; it then follows that these three virtual centres must be in a straight line.

To prove this, suppose that *a*, *b*, *c*, Fig. 75, are three links, that $I_{ab}$ and $I_{ac}$ are given, and that $I_{bc}$, instead of being on the line joining them, is at a point P.

Then P may be regarded as a point in *b*, and *b* rotates relatively to A at the given instant about $I_{ab}$, so that P

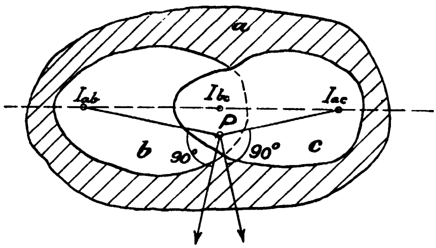


FIG. 75.

moves at right angles to $PI_{ab}$; similarly, regarded as a point in C, P moves about the point $I_{ac}$, and so moves at right angles to $PI_{ac}$. The only condition under which P can possibly move at the same time at right angles to $PI_{ab}$ and $PI_{ac}$ is that P should lie upon the line $I_{ab}$, $I_{ac}$.

If a mechanism has *n* links, there will be $\frac{n(n-1)}{2}$ instantaneous centres.

*General Case of Determination of Static Forces from Instantaneous Centres.*—Let *a*, *e*, Fig. 76, be any two links of a mechanism, between which there may be any number of intermediate links, and let *f* be the fixed link, and let the virtual centres $I_{af}$, $I_{ae}$, and $I_{ef}$ be known; these must be in a straight line. And let $F_a$ and $F_e$ be forces acting upon these links which are statically equivalent; then $F_e$ will be

equal and opposite to the force in the same direction required to keep the mechanism in equilibrium.

Take any convenient point H upon the line of action of $F_a$ and join H to $I_{af}$ and $I_{ae}$, and resolve the force $F_a$ in components $q$ and $p$ in these two directions; this is done by setting out 1, 2 to a convenient scale to represent $F_a$, drawing 1, 3 parallel to $H I_{ae}$ and 2, 3 parallel to $H I_{af}$. Now $I_{af}$ is a point upon the fixed link, it being remembered that a

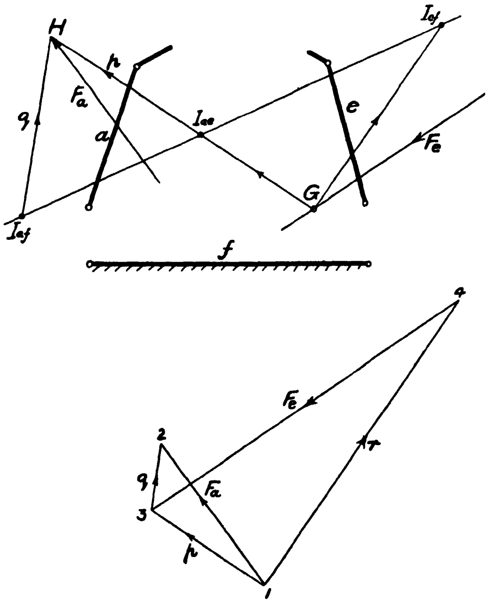


FIG. 76.

link is not necessarily a straight rod such as we show for convenience in our diagrams, and so the force $q$ acting through it can have no effect in causing relative movement of the movable links of the mechanism; we can, therefore, neglect $q$ in our further treatment. Next produce $H I_{ae}$ to meet the line of action of $F_e$ in G and join $G I_{ef}$; the force $p$ can then be resolved into the directions $F_e$ and $G I_{ef}$ by drawing 1, 4 parallel to $G I_{ef}$ and 3, 4 parallel to $F_e$, and

$F_e$ is given by the length 4, 3 to the force scale. The component $r$, acting towards the fixed point $I_{ef}$, can have no effect upon the movement of the links, and therefore the remaining component of $p$, viz. 3, 4, must be the force acting upon the link $e$, which is statically equivalent to the force $F_a$ acting upon the link $a$.

Considerable care in dealing with these problems is necessary to avoid treating them by the ordinary rules of resolution of forces employed in dealing with fixed bodies; such

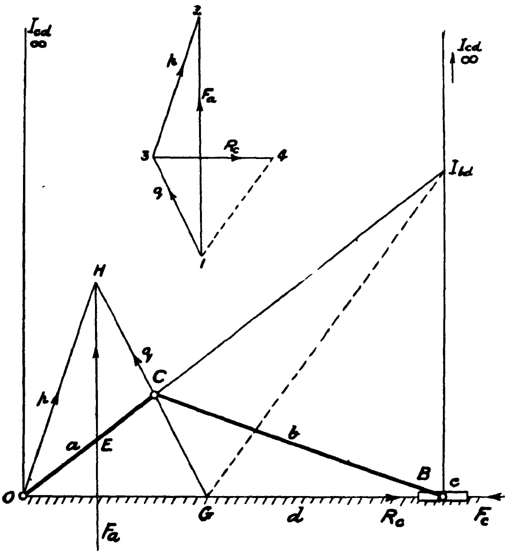


FIG. 77.

treatment often leads to error, as we will demonstrate later in dealing with the application of these principles to the quick-return mechanism used in shaping machines.

*Application to Crank and Connecting-rod Mechanism.*—In this case, Fig. 77, we have four links: the crank $a$, connecting-rod $b$, cross-head $c$, and the fixed framework of the mechanism $d$. By our formula there will be $\frac{4 \cdot 3}{2} = 6$ instantaneous centres, and by inspection we can mark the

fixed centres $I_{ad}$, $I_{ab}$, $I_{bc}$, because these are pivot points; the cross-head $c$ moves in a straight line along the frame $d$, and so their instantaneous centre $I_{cd}$ may be regarded as at infinite distance along a line at right angles to $d$, this being indicated by the sign $\infty$. Now $I_{ad}$, $I_{ab}$, and $I_{bd}$ must be in a straight line, and so must $I_{bc}$, $I_{bd}$, and $I_{cd}$; so, by joining $I_{ad}$, $I_{ab}$, and producing until it meets $I_{bc}$, $I_{cd}$, we get $I_{bd}$ at the intersection. The only remaining one is $I_{ac}$; to fix this we note that it must be in a line with $I_{ab}$, $I_{bc}$, and also with $I_{ad}$, $I_{cd}$. Now $I_{cd}$ is at infinite distance away, so draw a line through $I_{ad}$ parallel to $I_{bc}$, $I_{bd}$, and join $I_{bc}$, $I_{ab}$, producing to get the intersection at $I_{ac}$.

Now suppose, as the most general case, that a force $F_a$ acting upon the link $a$ intersects it at a point E, and that we require to find the horizontal resistance $R_c$ required for static equilibrium.

Take any point H on the line of action of $F_a$ and join H O, H C; then, by setting out 1, 2 to represent $F_a$ and drawing 1, 3; 2, 3 parallel to C H, O H, we get the components $p$, $q$ of $F_a$ in the given directions. The component $p$ acts towards the fixed point O and has no effect upon the force $R_c$. Now produce H C to meet the line of action of $R_c$ in G; then $q$ and $R_c$, acting upon the link $b$, keep it in equilibrium, and since this link is moving for the instant about the point $I_{bd}$, the resultant of $q$ and $R_c$ must pass through $I_{bd}$. Therefore draw 1, 4 parallel to G $I_{bd}$ and 3, 4 parallel to $R_c$ to meet in 4; then $R_c = 3, 4$.

In the usual case of this mechanism, $F_a$ acts at C at right angles to O C; we can then take moments about $I_{bd}$, and we shall have

$$F_a \times C I_{bd} = R_c \times B I_{bd}.$$

$$\therefore R_c = \frac{F_a \times C I_{bd}}{B I_{bd}}$$

$$= \frac{F_a \times O C}{O I_{ac}}.$$

This is, of course, the result which we should obtain by

the method of virtual velocities, because we have previously shown that

$$\frac{V_C}{V_B} = \frac{OC}{OI_{ac}}.$$

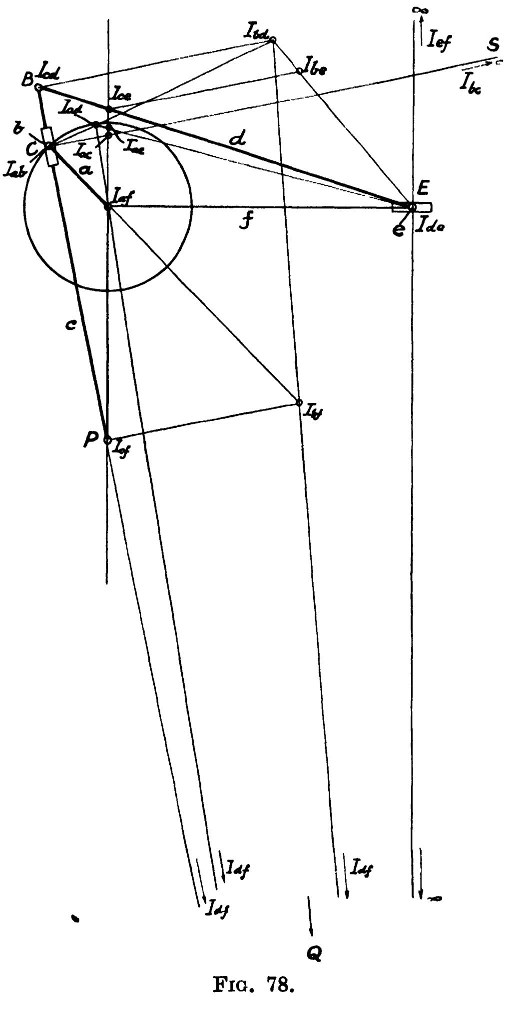


FIG. 78.

*Application to Quick-return Motion.*—We do not often require to find all the instantaneous centres for a mechanism,

but it is interesting to trace them out, and we will do so for the quick-return motion, which we have already considered.

Referring to Fig. 78, there are six links, viz. the crank $a$, the sliding block $b$, the slotted lever $c$, the connecting-rod $d$, the tool ram $e$, and the fixed frame $f$ of the machine. It should be remembered that, although we represent the links by straight lines, they may have any form, and in the case of the link $f$ we represent it by two straight lines $I_{af}E$, $I_{af}P$. This analysis of a mechanism into skeleton form assists materially in the study of its action. The number of instantaneous centres will be $\frac{6 \times 5}{2} = 15$.

By inspection we can insert the fixed centres $I_{cf}$, $I_{af}$, $I_{ab}$, $I_{cd}$, and $I_{de}$; then, since the tool ram $e$ moves horizontally along the frame, $I_{ef}$ will be at infinite distance along the vertical through E. Similarly, $I_{bc}$ will be at infinite distance along the line C S at right angles to B P, and we will remind the reader that a point at infinite distance away may be considered to be in either direction, as may be most convenient. We next fix $I_{df}$ by the fact that $I_{cd}$, $I_{cf}$, and $I_{df}$ are in a straight line, and $I_{de}$, $I_{ef}$, and $I_{df}$ are also in a straight line; so, by joining $I_{cd}$, $I_{cf}$, and producing to meet the vertical through E in Q, the point Q will be $I_{df}$. $I_{ac}$ will be in a line with $I_{af}$ and $I_{cf}$, and also in a line with $I_{ab}$ and $I_{bc}$; so draw through C a line parallel to C S (parallel lines meet at infinite distance) and produce $P\,I_{af}$ to meet it in $I_{ac}$. $I_{ce}$ will be in a line with $I_{cd}$, $I_{de}$, and also with $I_{cf}$, $I_{ef}$, and the last-mentioned will be at infinite distance along $P\,I_{af}$, so that $I_{ce}$ is at the intersection of B E and $P\,I_{af}$ produced. Next fix $I_{bf}$ by producing $C\,I_{af}$ to intersect a line through P parallel to C S, and join $Q\,I_{bf}$ and produce to meet a line through B parallel to C S to fix $I_{bd}$.

This leaves $I_{ad}$, $I_{ae}$, and $I_{be}$ still to be found. Joining $C\,I_{bd}$ and $Q\,I_{af}$ and producing, we get the intersection which fixes $I_{ad}$; then join this to E, the intersection with $P\,I_{af}$ produced giving $I_{ae}$. The final instantaneous centre $I_{be}$ is

obtained by joining $I_{bd}E$ and drawing a line through $I_{ce}$ parallel to CS. Now let us apply these results to find the force upon the crank-pin C equivalent to a force of 1,200 lb. acting upon the tool slide at E.

Now the point $I_{be}$ is rather awkward to find accurately, because we have to find the intersection Q; so we will first find the force $F_D$, Fig. 79, acting on the lever $c$, which is

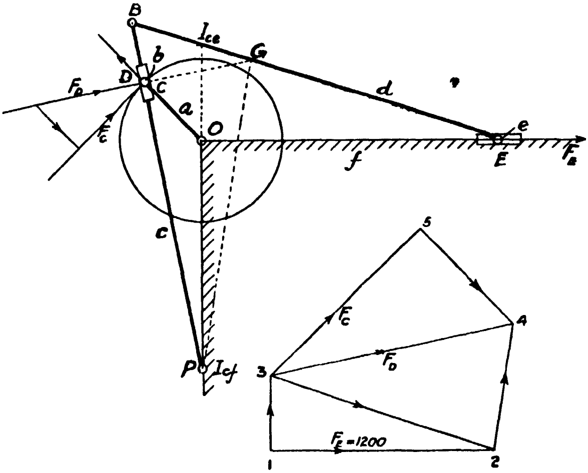


FIG. 79.

equivalent to $F_E$. The points $I_{ce}$, $I_{cf}$ are given, and $I_{ef}$ is at infinite distance along the vertical through E. Set out 1, 2 to represent $F_E = 1{,}200$ lb. and draw 1, 3 vertically (*i.e.* in the direction $E\,I_{ef}$) and 2, 3 parallel to $d$ (*i.e.* in the direction $E\,I_{ce}$). Let $F_D$ meet $d$ in G and then draw 2, 4 parallel to $G\,I_{cf}$ and 3, 4 parallel to $F_D$. This gives us $F_D$.

Now apply the same method to obtain $F_C$. $I_{bf}$ is on CO produced, so $F_D$ must be resolved in the direction of CO and parallel to $F_C$ to give $F_C$. Therefore draw 4, 5 parallel to CO and 3, 5 parallel to $F_C$; then $3, 5 = F_C$, which comes about 1,140 lb., as before.

We have here the example that we referred to previously

of the care which must be taken to avoid the ordinary ideas of resolution of forces for fixed bodies. Upon these ordinary ideas we should probably say that to find $F_C$ we should have to make the component of $F_C$ perpendicular to D equal to $F_D$, *i.e.* we should resolve $F_C$ along and at right angles to the slot D and should obtain a very incorrect result.

# CHAPTER VIII

## ELEMENTS OF FLY-WHEEL DESIGN

WE have seen that, in considering the motion of bodies from the standpoint of graphics, it is usual to plot the effort or propelling force against the distance moved, and to call the resulting diagram the *effort diagram*; on the same base may be plotted a diagram of the force resisting movement of the body, the resulting diagram being called the *resistance diagram*. The areas of these diagrams represent work or energy, and if the effort and resistance diagrams for a body do not coincide, a resulting amount of energy must be given to or taken from the body, this change in energy causing a change in speed. To keep this change in speed as small as possible, fly-wheels are employed; the fly-wheel stores up the increase in energy while the effort is greater than the resistance, and gives it out again when the resistance becomes greater than the effort.

The kinetic energy (K.E.) of a body of weight W lb. and radius of gyration $k$ ft.* rotating at $n_1$ revs. per sec. is given by the formula

$$\text{K.E.} = \frac{\text{W}\, 2\pi^2 k^2 n_1^2}{g} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

If the speed is changed to $n_2$ revs. per sec., the kinetic energy changes to $\frac{\text{W}\, 2\pi^2 k^2 n_2^2}{g}$, so that we have

$$\text{Change in K.E.} = e = \frac{\text{W}\, 2\pi^2 k^2}{g}(n_2^2 - n_1^2) \quad \ldots\ldots (2)$$

* A note on the radius of gyration is given at the conclusion of this chapter.

For a fly-wheel in which nearly all the weight is in a comparatively thin rim, $k$ will be practically equal to the radius, and for a fly-wheel in the form of a solid disc, $k$ will be equal to 0·707 time the radius; in an actual fly-wheel, therefore, the radius of gyration will be somewhere between 0·7 and 1 time the radius.

If W $k^2$, the moment of inertia of the fly-wheel, is large, it is clear from equation (2) that for a given change in kinetic energy the value of $(n_1^2 - n_2^2)$ will be small, *i.e.* the change in speed will be small.

**Coefficient of Fluctuation of Velocity.**—In the design of fly-wheels it is common to specify a certain "coefficient of fluctuation of velocity." This quantity, to which we will give the letter $q$, is defined by the relation

$$q = \frac{n_2 - n_1}{n_0}, \quad \ldots\ldots\ldots\ldots\ldots (3)$$

where $n_2$ = maximum speed of engine shaft; $n_1$ = minimum speed of engine shaft; and $n_0$ = mean speed of engine shaft.

Rankine called this quantity the "coefficient of unsteadiness"; it gives the proportional change of speed allowed, and its value depends upon the nature of the machine under consideration, but for engines in which smooth running is necessary a coefficient of 0·02 is commonly adopted.

The following values are given by Unwin and Mellanby in their *Elements of Machine Design*:

| | $n =$ |
|---|---|
| Engines doing pumping . . . . . . | $\frac{1}{20}$ |
| ,, driving machine tools . . . . . | $\frac{1}{35}$ |
| ,, ,, textile machines . . . . . | $\frac{1}{40}$ |
| ,, ,, spinning machinery . . . | $\frac{1}{50}$ to $\frac{1}{100}$ |
| ,, ,, electric machinery . . . . | $\frac{1}{150}$ |
| ,, ,, electric machinery (direct-driven) | $\frac{1}{300}$ |

Referring to Fig. 80, let A C B represent the *crank-effort diagram* of a double-acting steam-engine, *i.e.* a curve representing the force acting at right angles to the crank plotted against the distances moved through by the crank. The

crank effort is the resultant of the static force, transmitted to the crank by the steam pressure on the piston, and the inertia force necessary to bring about the acceleration of the reciprocating parts. The curve D E represents the resistance to the movement of the crank-pin caused by the machines which the engine has to drive. In engine design this resistance R is commonly nearly constant, so that the curve D E becomes a straight line.

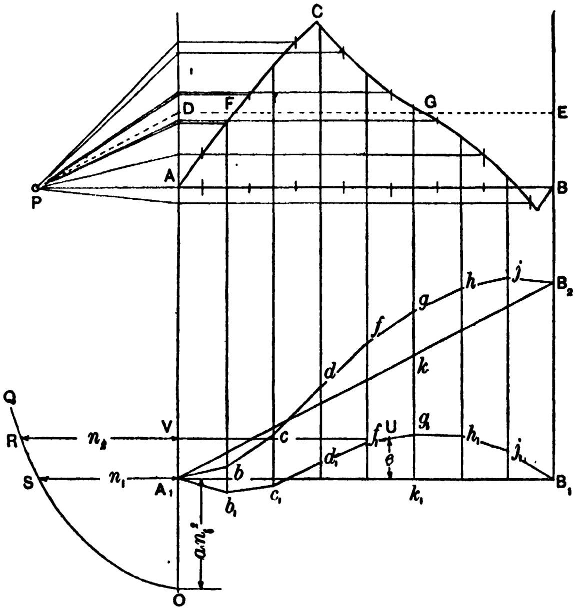


FIG. 80.—Fly-wheel Design.

If I.H.P. is the indicated horse-power of the engine, $r$ ft. the radius of the crank, and $n_0$ revs. per sec. the mean speed of the fly-wheel, we shall have

$$\text{Work done per sec.} = 550 \text{ I.H.P.-ft.-lb.}$$
$$= R \times 2\pi r n_0.$$
$$\therefore R = \frac{\text{I.H.P.}}{550 \times 2\pi r n_0} \quad \ldots\ldots\ldots\ldots \quad (4)$$

From the point A to F the resistance is greater than the effort, and so the fly-wheel has to supply an amount of energy represented by the area A D F, and the engine will slow down slightly; between the points F and G the effort is greater than the resistance, and so the fly-wheel has to store up an amount of energy represented by the area F C G, the engine thus increasing slightly in speed. For the remainder G B of the stroke the resistance is again greater than the effort, so that the fly-wheel has again to give out energy, and the engine again decreases slightly in speed.

If the engine is to run at a constant mean speed, *i.e.* if it is not gradually to increase or decrease in speed after a number of strokes, the speed at the beginning and the end of the stroke must be the same. The kinetic energy must, therefore, be the same at the beginning and at the end of the stroke, so that since the total change in kinetic energy between two points is represented by the difference in area of the effort and resistance curves, the areas of the two curves from A to B must be the same.

$$\therefore \text{Area A C B} = \text{area A D E B}.$$

Subtracting the common part, we have

$$\text{Area F C G} = \text{area A D F} + \text{area G E B}.$$

The height of the line D E, to fulfil this condition, can be found graphically, as described later.

The velocity of the fly-wheel will have a minimum value at the point F and a maximum value at the point G, because from the point corresponding to G in the previous stroke to the point F it has been slowly decreasing; it then increases slowly until the point G is reached, at which the whole of the excess energy has been absorbed.

We have assumed that the crank effort is the same during the outstroke as during the instroke, but, except in the case of an engine with a tail rod equal in area to the piston-rod, the effort will be greater on the outstroke than on the instroke on account of the loss of pressure caused by the

piston-rod. In this case we must draw the effort and resistance curves for a complete revolution, and the total positive area must be equal to the total negative area.

Similarly, in internal-combustion engines in which two revolutions are required to complete the cycle we must draw the curves for two revolutions.

**Coefficient of Fluctuation of Energy.**—Let the energy represented by the area FCG be $e_n$, and let the total work done in one stroke, *i.e.* the area ADEB or ACB, be $E_n$. Then the ratio $\frac{e_n}{E_n}$ is called the "coefficient of fluctuation of energy," and is given the letter $m$.

This coefficient will depend both upon the kind of engine employed and upon the manner in which the resistance varies. Assuming that the resistance is constant, it is common to take $\frac{e_n}{E_n}$ for steam-engines as 0·25 for single-cylinder engines, 0·18 for compound engines, and 0·12 for two-cylinder engines with cranks at right angles. An average value for single-cylinder internal-combustion engines working on the four-stroke cycle is 3·5.

Returning to equation (2), we have

$$e_n = m\,E_n = \frac{W\,2\pi^2 k^2}{g}(n_2{}^2 - n_1{}^2)$$

$$= \frac{W\,2\pi^2 k^2}{g}(n_2 - n_1)(n_2 + n_1);$$

but $\frac{(n_2 + n_1)}{2}$ = mean speed (approx.) $= n_0$; and by equation (3) $(n_2 - n_1) = q\,n_0$, so that we have

$$m\,E_n = \frac{W\,4\pi^2 k^2 n_0{}^2 q}{g} \quad \ldots\ldots\ldots\ldots\ (5)$$

From this the necessary weight W of a fly-wheel for any given conditions is determined by the relation

$$W = \frac{m\,E_n\,g}{4\pi^2 k^2 n_0{}^2 q};$$

and since

$$E_n = \frac{\text{I.H.P.} \times 550}{2\,n_0},$$

$$W = \frac{550\,g}{8\pi^2}\,\frac{m}{q}\,\frac{\text{I.H.P.}}{k^2\,n_0^3} \ldots\ldots\ldots\ldots \text{(6)}$$

$k$ being in feet, this gives

$$W = \frac{224\,m}{q}\,\frac{\text{I H.P.}}{k^2\,n_0^3}\ \text{lb.} \ldots\ldots\ldots \text{(7A)}$$

$$= \frac{m}{10\,q}\,\frac{\text{I.H.P.}}{k^2\,n_0^3}\ \text{tons} \ldots\ldots\ldots \text{(7B)}$$

If, instead of a mean speed of $n_0$ revs. per sec., we have one of $N_0$ revs. per min., we shall have

$$W = 48{,}400{,}000\,\frac{m}{q}\,\frac{\text{I.H.P.}}{k^2\,N_0^3}\ \text{lb.} \ldots\ldots\ldots \text{(8A)}$$

$$= 21{,}600\,\frac{m}{q}\,\frac{\text{I.H.P.}}{k^2\,N_0^3}\ \text{tons} \ldots\ldots\ldots\ldots \text{(8B)}$$

NUMERICAL EXAMPLE.—*Find the necessary weight of the fly-wheel of a* 120-*h.p. steam-engine running at* 80 *revs. per min., if the radius of gyration is* 5 *ft. and the fluctuation of energy is* 12% *per stroke. Take a fluctuation of velocity of* 2% *on either side of the mean.*

In this case the total fluctuation of velocity is 4%, *i.e.* $q = 0{\cdot}04$ and $m = 0{\cdot}12$. Therefore, by equation (8B), we have

$$W = 21{,}600 \times \frac{0{\cdot}12}{0{\cdot}04} \times \frac{120}{5 \times 5 \times 80 \times 80 \times 80}\ \text{tons}$$

$$= 0{\cdot}61\ \text{ton.}$$

**Graphical Investigation of the Variation of Velocity.**—To find graphically the line D E of mean resistance, and to make a complete study of the variation of velocity over the stroke, we must first draw the curve of work by drawing the sum curve of the effort curve, thus obtaining the curve $A_1\,b\,c\,d\,f\ \ldots\ B_2$, Fig. 80.

The work done by the effort must be equal to that done against the resistance by the end of the stroke, so that we join $A_1\,B_2$ and reduce the curve to a horizontal base $A_1\,B_1$

by setting up a length $k_1 g_1$ equal to $k g$, and so on for the various ordinates of the curve upon the inclined base $A_1 B_2$. To obtain the line D E of mean resistance we then draw through the pole P of the sum curve construction a line P D parallel to $A_1$ $B_2$; then the line D E gives the line of mean resistance required. If the resistance is not constant, but varies according to any given law, such as that indicated

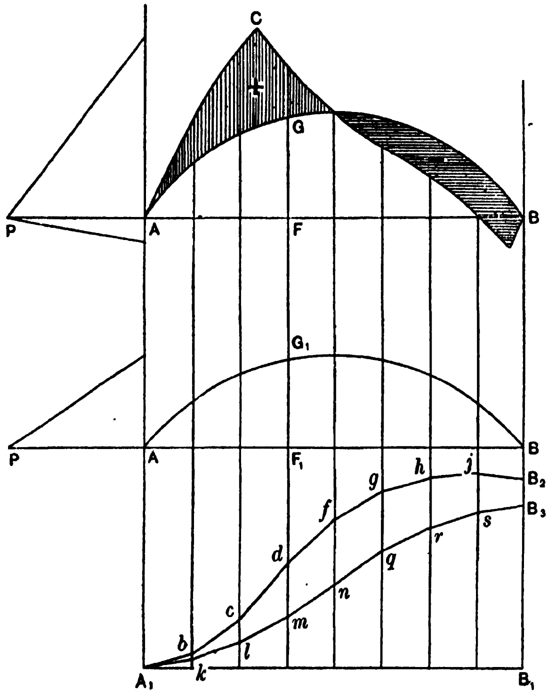


FIG. 81.

by the curve A G B in Fig. 81, we still have the condition that the area of the crank effort curve and the resistance curve must be the same. If we know the manner in which the resistance varies, but do not know the exact curve to fulfil the above condition, we proceed as follows : First draw a trial or assumed resistance curve A $G_1$ B, and find the

sum curve $A_1\,b\,c\,\ldots\,B_2$ of the crank effort curve, and also that, $A_1\,k\,l\,m\,\ldots\,B_3$, of the assumed resistance curve.

Take any ordinate $F_1\,G_1$ of the assumed resistance curve and calculate $F\,G$ such that

$$F\,G = \frac{F_1\,G_1 \times B_1\,B_2}{B_1\,B_3}.$$

Do this for each ordinate, and the resulting curve $A\,G\,B$ will be the resistance curve required.

*Auxiliary Parabola Construction.**—To study the variation in velocity we may use the above construction. Suppose that at any point—say, at $A_1$, Fig. 80, the beginning of the stroke—the number of revolutions per sec. is $n_1$, and that at some other point U the increase in kinetic energy, *i.e.* the ordinate of the curve of work, is $e$. Then we have by equation (2)

$$e = \frac{W\,2\pi^2\,k^2}{g}\,(n_2{}^2 - n_1{}^2).$$

Calling $\dfrac{W\,2\pi^2\,k^2}{g} = a$, we have

$$e = a\,(n_2{}^2 - n_1{}^2),$$

*i.e.*
$$n_2{}^2 = \frac{e}{a} + n_1{}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

Now set down $A_1$ O equal to $a\,n_1{}^2$ to the scale to which the curve of work is drawn, and choosing a convenient scale draw $A_1$ S to represent $n_1$, and with vertex at O draw a parabola O S Q to pass through the point S; then, by projecting any point such as U on to this auxiliary parabola, we get V R, representing $n_2$, the number of revolutions per sec. at the given point which has been chosen. For by the property of the parabola

$$\frac{R\,V^2}{O\,V} = \frac{S\,A_1{}^2}{O\,A_1},$$

*i.e.*
$$\frac{R\,V^2}{a\,n_1{}^2 + e} = \frac{n_1{}^2}{a\,n_1{}^2} = \frac{1}{a}.$$

* See p. 38 for this construction in the general case.

$$\therefore \ \mathrm{R}\,\mathrm{V}^2 = \frac{a\,n_1^2 + e}{a}$$

$$= \frac{e}{a} + n_1^2 = n_2^2 \text{ [by equation (9)].}$$

**Energy Stored in Fly-wheel in Terms of Strokes.**—A very useful way of stating the energy to be stored in a fly-wheel can be obtained as follows:

The work done per stroke is given by

$$\mathrm{E} = \frac{550 \text{ I.H.P.}}{2\,n_0}.$$

Further, the mean energy stored in the fly-wheel is equal to

$$\frac{\mathrm{W}\,2\pi^2 k^2 n_0^2}{g}.$$

$\therefore$ Number of strokes of work stored

$$= \frac{\mathrm{W}\,2\pi^2 k^2 n_0^2}{g} \div \frac{550 \text{ I.H.P.}}{2\,n_0}$$

$$= \frac{0{\cdot}00223\,\mathrm{W}\,k^2 n_0^3}{\text{I.H.P.}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

From the equation in this form we can find the weight of fly-wheel of given radius of gyration required for an engine in which the number of strokes to be stored is given.

In punching, shearing, and slotting machines, for instance, it is common to design the fly-wheel so that it has stored in it the energy of two working strokes.

If we are given the coefficients of fluctuation of velocity and energy ($q$ and $m$ respectively), we have, by putting in the value of W from equation (7A),

Number of strokes stored

$$= \frac{0{\cdot}00223\,k^2 n_0^3}{\text{I.H.P.}} \frac{224\,m}{q} \frac{\text{I.H.P.}}{k^2 n_0^3} = \frac{m}{2\,q}.$$

For given values of $m$ and $q$ we therefore have a simple calculation for finding the corresponding number of strokes to be stored.

**Note on the Radius of Gyration.**—To calculate the radius of gyration of a solid body about an axis of rotation,

we must first imagine the body to be divided up into an infinitely large number of extremely narrow threads drawn parallel to the axis. If $w$, Fig. 82, is the weight of one of these elementary threads, and $r$ is its distance from the axis, we multiply $w$ by $r^2$ and add together the results of the separate calculations for all the threads into which we have divided the body; then the result is called the moment of inertia of the whole body about the given axis or pole. This is expressed in symbols by

$$\mathrm{I} = \text{moment of inertia of body} = \text{sum of quantities like } w r^2 = \Sigma (w r^2).$$

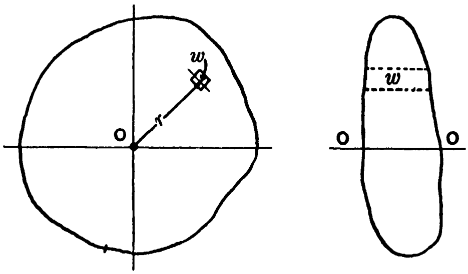


FIG. 82.

Then if $k$ is the radius of gyration of the body and W its weight, we write

$$\mathrm{W} k^2 = \mathrm{I},$$

*i.e.*
$$k = \sqrt{\frac{\mathrm{I}}{\mathrm{W}}}.$$

Thus the radius of gyration may be regarded as the radius at which the weight of the whole body may be considered concentrated so as to give the same moment of inertia as that obtained by dividing the body up into small threads.

If the body is of such form that its outline can be expressed by means of a formula, the summation may be found mathematically by means of the integral calculus.

In practical sections which do not admit of this method, we may, if great accuracy is required, divide up into a great number of threads and add the results or employ a graphical

construction, such as that described in the author's *Theory and Design of Structures*; but few calculations require this accuracy. For most cases of fly-wheel with arms it is accurate enough to take

$$W k^2 = \left(W_r + \frac{W_a}{3}\right) R^2,$$

where $W_r$ = weight of rim, $W_a$ = weight of arms, R = radius (feet) to the centroid of the rim.

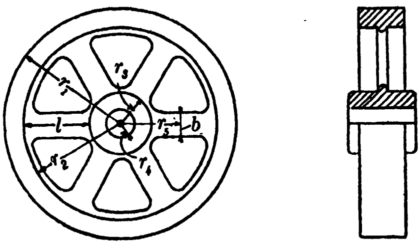

Fig. 83.

A more accurate, but still strictly approximate, method is as follows (see Fig. 83): Let $W_r$, $W_a$, and $W_b$ be the weights of rim, arms, and boss respectively; then

$$W = W_r + W_a + W_b$$

and

$$W k^2 = W_r\left(\frac{r_1^2 + r_2^2}{2}\right) + W_b\left(\frac{r_3^2 + r_4^2}{2}\right) + W_a\left(\frac{l^2 + b^2}{12} + r_5^2\right).$$

For a solid disc fly-wheel of radius $r$, $W k^2 = \dfrac{W r^2}{2}$.

# CHAPTER IX

## BALANCING ROTATING PARTS

WHEN a body rotates about a fixed centre at constant speed its velocity is continually changing in direction, though not in magnitude; any continuous change in velocity, whether in direction or in magnitude, or in both, indicates that the body is receiving an acceleration which, in accordance with Newton's laws of motion, must be caused by an external force. In the case of a body rotating in a circular path at constant speed it is known that the resulting acceleration is towards the centre, and is called the *centripetal acceleration.* The body tends to fly out at a tangent, and a force acts radially outwards which is equal and opposite to that causing the centripetal acceleration. This is called the *centrifugal force.*

We have shown on p. 93 that the value of the centrifugal force $F_c$ is given by the formula

$$F_c = \frac{W v^2}{g r}, \qquad (1)$$

$g$ being the gravity acceleration of 32·2 ft. per sec. per sec.

If, as is usual, the speed is measured in revolutions per unit time—say, $n$ revs. per sec.—we see by the rule that in 1 rev. the point A moves through $2\pi r$ ft., that in 1 sec. the body moves through $2\pi r n$ ft., so that since $v$ is also the distance moved through in 1 sec., $v = 2\pi r n$.

$$\therefore \ F_c = \frac{W (2\pi r n)^2}{g r}$$

$$= \frac{4\pi^2 n^2 r W}{g} \qquad (2)$$

Since this force is proportional to the square of the speed, it will easily be seen that at high speeds the centrifugal force becomes large ; and if the weight is attached to a wheel or drum rotating at high speed, the effect of this force will be to set up vibrations in the shaft, and a knocking tendency on the bearings, which will interfere greatly with the even running of the machine, and in the case of large engines such as are employed in electric generating stations it may cause a vibration in all the buildings in the neighbourhood. In motor-cars and marine engines even running is also essential. The procedure for preventing the existence of the centrifugal forces which are the cause of the uneven running is called *balancing.*

*A shaft, pulley, or drum is said to be perfectly balanced when the centrifugal forces due to the various portions of which it is composed neutralise each other completely, and so have no resultant. In the language of graphics, therefore, our first condition for balance is that the vector diagram for the centrifugal forces must be a closed polygon.*

**A. Centrifugal Forces in the Same Plane.**—If all the centrifugal forces to be considered are in the same plane, the above condition will be the only one to be considered, and so we may at once pass on to a consideration of some of the special cases which arise.

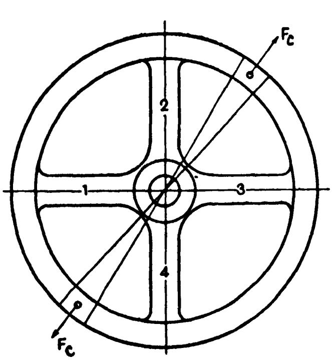

FIG. 84.

*Case A*1 : *Symmetrical Body.*—By this we mean a body, such as the pulley shown in Fig. 84, in which a centre line can be drawn which divides the body into two portions which are exactly similar and may be regarded as mirror-reflections of each other, the centre line representing the mirror.

From a study of Fig. 84 it will be seen that the centrifugal

force due to the opposite arms 1, 3 and 2, 4 will be equal and opposite, and will, therefore, balance each other; and if we consider any two portions of the rim enclosed by two diameters a small angular distance apart as shown, we shall see that the centrifugal forces F for these portions will be equal and opposite and balance each other, and that, since the whole rim may be thus divided into equal portions opposite to each other, the whole rim is balanced. This, of course, assumes that the rim is of the same dimensions

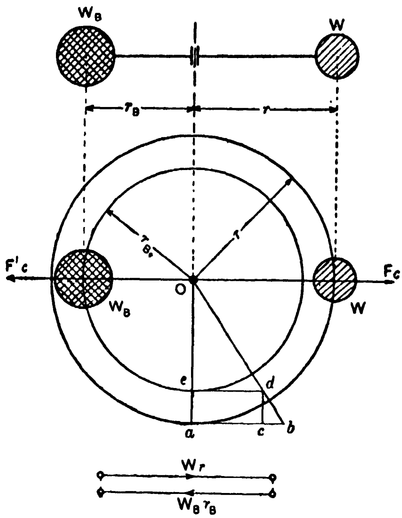


FIG. 85.—Balancing Single Eccentric Load.

throughout and is homogeneous, *i.e.* that it contains no blowholes or excrescences which will disturb its balance.

We see, therefore, that bodies which are symmetrical about a line through the axis of rotation are balanced.

*Case A2: Single Eccentric Load.*—Suppose that the body has a single eccentric load W, Fig. 85, and that we wish to determine where to place a second load $W_B$ so as to balance it. The load W induces a centrifugal force $F_c = \frac{4\pi^2 n^2 r}{g} \cdot W$,

and this is the only force tending to disturb the even running of the body.

Now a single force can be neutralised or balanced only by an equal and opposite force, so that we see, in the first place, that the balance-load $W_B$ must be placed at some radius $r_B$ diametrically opposite to W, and that its centrifugal force $F'_c$ must be equal to $F_c$.

We therefore have

$$F_c = \frac{4\pi^2 n^2 r W}{g} = F'_c = \frac{4\pi^2 n^2 r_B W_B}{g}.$$

The term $\frac{4\pi^2 n^2}{g}$ is common to the two forces, and will be common to all the centrifugal forces acting upon the same body. To save repetition, therefore, we will omit it from further cases and write our relation as

$$W r = W_B r_B,$$

*i.e.* $$r_B = \frac{W r}{W_B} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3A)}$$

or $$W_B = \frac{W r}{r_B} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3B)}$$

The radius $r_B$, at which the balance weight $W_B$ must be placed, may be determined graphically as follows:

Draw any radius O*a* and draw *ab* at right angles to it, and make *ab* equal to the weight $W_1$ on some convenient scale, and then make *ac* equal to W on the same scale; join O*b* and draw *cd* parallel to O*a*, and *de* parallel to *ab*; then O*e* gives the required radius $r_B$.

If, on the other hand, we know the position at which we wish to place the balance weight, and require to find graphically the value that it should have, we may modify the construction as follows:

Draw a circle through the centre of the weight W to be balanced, and a concentric circle of radius $r_B$, and draw any radius O*a*, cutting the balance weight circle in *e* (if $r_B$ is greater than *r*, *e* will be on O*a* produced); then draw *ed*

at right angles to $Oe$, and make $ed$ to represent W on some convenient scale, and join $Od$, producing it to meet in $b$ the line drawn through $a$ at right angles to $Oa$; then $ab$ gives the required value of $W_B$.

To prove this construction, we note that the triangles $Oed$, $Oab$ are similar, and that, therefore,

$$\frac{ab}{ed} = \frac{Oa}{Oe},$$

*i.e.*

$$ab = \frac{ed \times Oa}{Oe} = \frac{Wr}{r_B}.$$

Therefore, by equation (3B), $ab = W_B$.

To take a numerical illustration, let the weight W be 12 lb. acting at a radius of 15 in., and that we require to find the radius at which a balance weight of 20 lb. should be placed. We then have, by equation (3A),

$$r_B = \frac{Wr}{W_B} = \frac{15 \times 12}{20} = 9 \text{ in.}$$

The graphical construction will produce the same result.

*Case A3: Two or More Eccentric Loads.*—Next consider the case in which we have two loads $W_1$ and $W_2$, Fig. 86, rotating in the same plane at radii $r_1, r_2$ respectively, and that we wish to balance them by a single load $W_B$ at radius $r_B$.

As the system rotates we shall have three centrifugal forces induced, viz. $F_{c1}$, $F_{c2}$, and $F_{cB}$, and these have to be in equilibrium.

Numbering the spaces between the forces, and remembering that, as the term $\frac{4\pi^2 n^2}{g}$ will occur in each centrifugal force, we can represent the forces graphically by lines drawn parallel to them proportional to $W_1 r_1$, $W_2 r_2$, and $W_B r_B$ (the effect of multiplying each by $\frac{4\pi^2 n^2}{g}$ will only be to alter the scale), we then proceed as follows: Draw 1, 2 on the vector diagram parallel to $OW_1$ to represent $W_1 r_1$; then draw 2, 3 parallel to $OW_2$ to represent $W_2 r_2$ on the same scale

and join 3, 1. Then 3, 1 represents in magnitude, direction, and sense the value that $W_B r_B$ must have, and the angle $\theta$ gives the angular position with regard to O $W_1$ of the radius upon which the balance weight $W_B$ must be placed.

If we wish to obtain by graphical construction the value of $W_B$ which is required when the radius $r_B$ is given, we may draw 1 *a* in a convenient direction to represent $r_B$ to a suitable

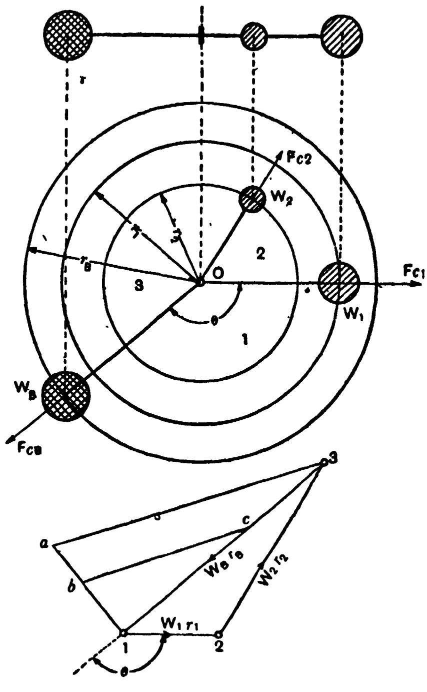


FIG. 86.—Balancing Two Rotating Weights.

scale, and make 1 *b* represent unit length to the same scale; we then join *a* 3 and draw *b c* parallel to *a* 3 to cut 1, 3 in *c*; then 1 *c* measured on the scale of the vector diagram will represent $W_B$. The construction can be varied in a manner which will be obvious from a study of Fig. 86 when the balance weight is given and we require to know the radius at which it should act.

If we have more than two loads, we proceed in exactly the same manner, except that instead of a triangle for the vector figure we shall have a polygon; in every case the line joining the last point to the first point of the vector polygon gives in magnitude the value of $W_B r_B$ and the direction of the radius on which the balance weight should lie.

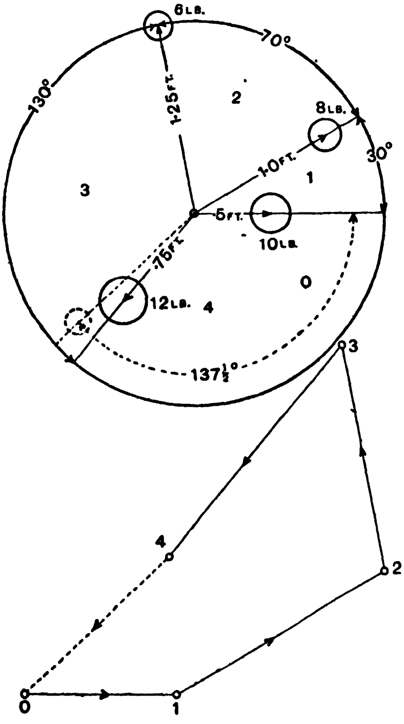


FIG. 87.—Numerical Example of Balancing.

This should be made clear from the following numerical example.

NUMERICAL EXAMPLE.—*Four weights of magnitude* 10, 8, 6, 12 *lb., and radii* 0·5, 1·0, 1·25, 0·75 *ft. respectively, rotate at* 200 *revs. per min. at the relative angular positions shown in full lines in Fig.* 87. *Find the magnitude and angular*

*position of the single force required to balance the system, and find also the resulting force on the shaft if the balance weight were not provided.*

We will first tabulate as follows :

| No. | Weight in lb. W. | Radius in ft. $r$. | W $r$ in lb.-ft. |
|---|---|---|---|
| 1 | 10 | 0·5 | 5 |
| 2 | 8 | 1·0 | 8 |
| 3 | 6 | 1·25 | 7·5 |
| 4 | 12 | 0·75 | 9 |

Then we number the spaces between the various radii, and choosing a suitable scale, we draw 0, 1 horizontally to represent 5 units ; we then draw 1, 2 parallel to the radius of the weight 2 and make it represent 8 units to the same scale ; then 2, 3 parallel to the radius of weight 3 to represent 7·5 units ; and 3, 4 for weight 4 to represent 9 units.

Then 4, 0 gives us the required value of $W_B\, r_B$, and will be found to scale 6·6 units ; as, therefore, our balance weight has to be placed at 1 ft. radius, its value must be 6·6 lb.

On drawing, as shown in dotted lines, a radius parallel to 4, 0, and in the direction 4 to 0, we find that the angular position of this balance weight must be $137\frac{1}{2}°$ approx. to the first weight.

To calculate the force on the shaft if the balance weight were not provided, we note that the three weights at their respective radii have been shown to be equivalent to a single weight of 6·6 lb. at 1 ft. radius. By equation (2) we have

$$F_c = \frac{4\pi^2 n^2 r\,W}{g},$$

and we are given that the shaft goes at 200 revs. per min., *i.e.*

$$n = \frac{200}{60} = \frac{10}{3}.$$

$$\therefore\ F_c = \frac{4 \times \pi^2 \times 10^2 \times 1 \times 6{\cdot}6}{9 \times 32{\cdot}2}$$

$$= 90 \text{ lb. approx.}$$

**Testing Balance of Weights in One Plane; Standing Balance.**—If an unbalanced wheel or other body having weights in the same plane is mounted on a shaft, and the shaft is laid across two knife-edges, the shaft will roll until the heavy side of the wheel comes to the bottom. This can be seen very well by lifting the front wheel of a bicycle the valve of which throws the wheel out of balance; the wheel will oscillate, and will ultimately come to rest with the valve near the bottom.

For this reason the balance obtained by considering weights in one plane is called a *standing balance*; a more delicate method of testing is to mount the wheel and shaft with driving gear upon a bed mounted upon springs, and any lack of balance will be shown by the bed vibrating upon its springs. This is called a *running balance* test. For weights in the same plane a system which is proved to be balanced by a standing test should also prove to be balanced by a running test, but when the weights are not in the same plane, *i.e.* in a plane at right angles to the axis of rotation, the two tests will not necessarily give the same result, and so both tests should be applied.

The difference between the results of the standing and the running tests of weights not in the same plane is due to the existence of centrifugal "couples" as well as centrifugal forces, which we have next to consider.

B. **Centrifugal Forces Not in the Same Plane.**—We have considered up to the present only those cases in which the centrifugal forces due to the revolving parts are in the same plane; we now come to the much more difficult problem of balancing rotating parts which are in different planes.

Consider in the first place two equal weights W, Fig. 88, at the same radius $r$ mounted on parallel arms at opposite sides of a shaft and at a distance $a$ apart measured parallel to the axis of rotation. If the arms are parallel, they must lie in a common plane passing through the axis of rotation; it is easier to explain this statement by reference to a diagram

than to make it clear in words. The fine lines in Fig. 88 denote this plane, and we will use the term "elevation" to indicate a projection upon a plane at right angles to the axis of rotation, and the term "plan" to represent the projection upon a plane containing the axis; with this convention we can express the condition that, if two radii are parallel, they will be upon a common diameter in the elevation.

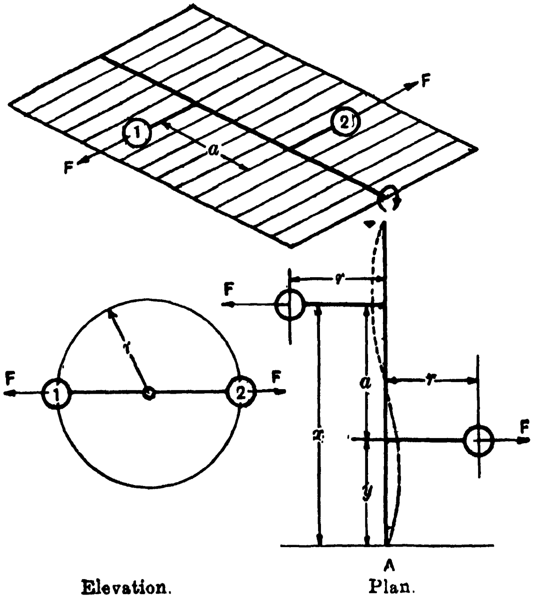


FIG. 88.—Balancing Centrifugal Forces not in the Same Plane.

Now in our case the centrifugal forces F at any instant will be equal and opposite, but they will not neutralise each other because they are in different planes, and so give rise to what is known as a *centrifugal couple*, which tends to give the shaft a wobbling motion. This will be clear if we imagine the shaft to be very flexible; then the effect of the equal and opposite forces F will be to make the shaft deflect at the instant under consideration into some form such as

shown in dotted lines in the plan, thus giving rise as the shaft revolves to the wobbling motion referred to above.

**Properties of Couples.**—It will help us to understand the problem of balancing more clearly if we make ourselves quite familiar with the properties of couples, especially couples in different planes. In all problems involving different planes there is some difficulty due to the fact that we cannot represent fully in two dimensions, such as we have in a sheet of paper, bodies having three dimensions.

(1) *Constancy of Moment about Any Point in the Plane of the Couple.*—In the simple case which we have considered we will take moments about the point A and will adopt the convention of calling a clockwise moment positive, and an anticlockwise moment negative; we then have

$$\text{Moment about A due to weight } 1 = -\mathrm{F}x.$$
$$\text{,, \quad ,, \quad ,, \quad ,, } \quad 2 = +\mathrm{F}y.$$
$$\therefore \text{ Resultant moment} = -\mathrm{F}x + \mathrm{F}y$$
$$= -\mathrm{F}(x - y)$$
$$= -\mathrm{F}a.$$

It will be seen that this resultant moment, called the moment of the couple, does not depend at all upon the position of the point A in the plane of the couple about which we took moments; this is in accordance with the property of couples that "the moment of a couple is the same about any point in its plane."

(2) *Resultant of Couples in Different Planes.*—Let one couple formed of forces $\mathrm{F}_1$ at arm $a_1$ act in a plane 1, Fig. 89, and let another couple composed of forces $\mathrm{F}_2$ at arm $a_2$ act in a second plane 2, the line X Y being the line of intersection of the two planes. Then on an "elevation" on a reference plane drawn through X at right angles to X Y the plane 1 is represented by X 1, and the plane 2 by X 2. Now regard the moments $\mathrm{F}_1 a_1$ and $\mathrm{F}_2 a_2$ of the couples as vectors acting along X 1 and X 2, and draw a vector diagram for the moments; this consists of the triangle

$b\,c\,d$, $b\,c$ representing $F_1\,a_1$ and $c\,d$ representing $F_2\,a_2$ to the same scale. The directions of the vectors $b\,c$, $c\,d$ are taken according to the following rule: If the moments of the couple are clockwise, we draw the vectors from left to right, or upwards; and if they are counterclockwise, we draw from right to left, or downwards. In our case, therefore, if the forces $F_2$ were in the reverse direction, we should draw downwards from $c$ in the vector figure, thus obtaining the diagram shown in dotted lines.

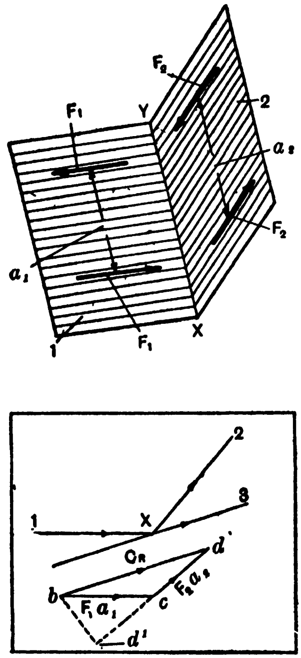

Fig. 89.—Couples in Different Planes.

Joining $b\,d$, we complete the moment triangle, and we obtain the magnitude and direction of plane of the resultant couple $C_R$, which, in accordance with our rule, will be clockwise.

If we had any number of couples in planes all intersecting in the line X Y, we should proceed in exactly the same way, but we should have a moment polygon instead of the moment triangle, and the resultant couple would be obtained by joining the first point in the polygon to the last; as in the case of the ordinary polygon of forces, it does not matter in what order we take our vectors. In all the problems of balancing which we shall consider the planes of all the couples will intersect in a common line—the axis of rotation—so that we will not consider further the case of couples in planes not intersecting in a common line.

**Application to Balancing Problems.**—Suppose that we have bodies 1, 2 of weight $W_1$, $W_2$ arranged as shown

in Fig. 90 at radii $r_1, r_2$ on a rotating shaft, and consider any convenient reference plane at right angles to the shaft passing through a point on the axis. Now imagine to act through this point on this reference plane equal and opposite forces $F_1$, $F_1$; $F_2$, $F_2$ parallel to the centrifugal forces $F_1$, $F_2$ due to the rotation of the masses 1, 2 respectively; these equal and opposite forces neutralise each other, and can

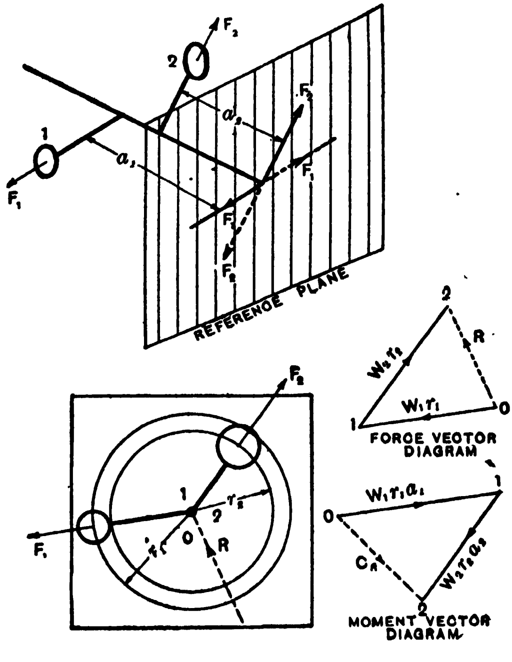


FIG. 90.—Balancing Weights in Different Planes.

have no effect upon the whole system of forces. We can now regard our forces $F_1$ and $F_2$ as equivalent to two forces $F_1$, $F_2$ in the reference plane, and two couples of moments $F_1 a_1$, $F_2 a_2$ acting in the planes containing the axis of rotation and the two arms $r_1$ and $r_2$ respectively. Since, as we have already shown, the centrifugal forces are proportional to the radius and to the square of the number of revolutions

per min., and the latter are the same for the whole system we may write

$$F_1 = W_1 r_1; \quad F_2 = W_2 r_2.$$

$$\therefore \; F_1 a_1 = W_1 r_1 a_1; \quad F_2 a_2 = W_2 r_2 a_2.$$

By means of the force vector diagram we can find the resultant force in the reference plane, and by means of the moment vector diagram we can find the resultant moment about the reference plane; these diagrams are shown in Fig. 90 and should be followed without difficulty. If we had any number of masses the same argument would apply, but we should have polygons instead of triangles for our vector diagrams.

**Conditions of Balance.**—Now if a rotating body is to be perfectly balanced there must be no resultant forces tending to lift the body right off its bearings, and there must be no resultant couples tending to give the shaft a wobbling motion; but we have seen that the resultant force is given by the closing line of the force vector diagram, and that the resultant couple is given by the closing line of the moment vector diagram, so that our conditions of balance become:

1. *The force vector diagram must close.*
2. *The moment vector diagram must close.*

Bodies which satisfy condition 1 only are said to be *statically balanced*, *i.e.* they will remain still in any position when freely supported in bearings at their ends. Bodies which satisfy both conditions are said to be *dynamically balanced.*

**Special Cases.**—(1) *Two Equal Weights at Equal Radii and Offset on Opposite Sides* (Fig. 88).—We have already considered this case in studying our first ideas of couples. Now let us see if we can bring this system into a state of perfect balance by means of a single balance weight placed at some suitable position; a little thought will show us that we cannot, because, no matter where we apply the third load, the first of the two conditions enumerated above will cease to hold, since the two forces F already have a zero

resultant, so that the resultant of the three centrifugal forces will be equal to that of the third weight.

Next let us try to bring these two masses into balance by applying two balance weights which we will call $W_{B1}$ and $W_{B2}$, placed at radii $r_1$ and $r_2$ respectively.

From what we have seen already it is clear that the centrifugal force due to these two weights must have a zero resultant, *i.e.* if they are considered as in the same plane normal to the axis of rotation, they must balance each other, and we have already seen that for this to be the case we must have

$$W_{B1} r_1 = W_{B2} r_2 = F_B.$$

Further, it is clear that the couple due to the forces F at arm $a$ can be neutralised only by a couple of reverse moment in the same plane. We have, therefore, the following conditions to satisfy : Two weights must be placed on opposite radii at a distance $b$ apart, such that

$$W_{B1} r_1 b = W_{B2} r_2 b = W r a,$$

*i.e.*

$$F_B b = F a,$$

*i.e.*

$$b = \frac{F a}{F_B}.$$

Fig. 91 shows one possible arrangement, but there are an infinite number of ways, since we may take any value and radius we like for one of the balance weights, the value and radius of the other and the distance apart of the weights being determined by the above formulæ. We now have two loads on each side of the shaft, and if we find the lines of action of the resultants of the centrifugal forces $F + F_B$ on each side, these lines of action should coincide as shown ; otherwise we should still have a resultant couple which would cause a lack of dynamic balance.

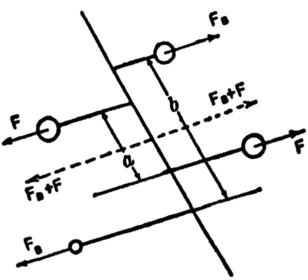


FIG. 91.

NUMERICAL EXAMPLE.—It may help to make more clear the conditions which we have explained up to the present by symbols if we work a numerical example.

*Find suitable balance weights for two weights each of* 10 *lb. placed at opposite radii of* 1 *ft. at a distance of* 9 *in. apart.* We can balance these by any two weights placed at opposite radii which are in the plane of the radii of the weight to be balanced, provided that they are placed at the correct distance apart. In general it is more convenient to use equal balance weights, and we will assume that convenient values will be balance weights of 4 lb. placed at a radius of 20 in.

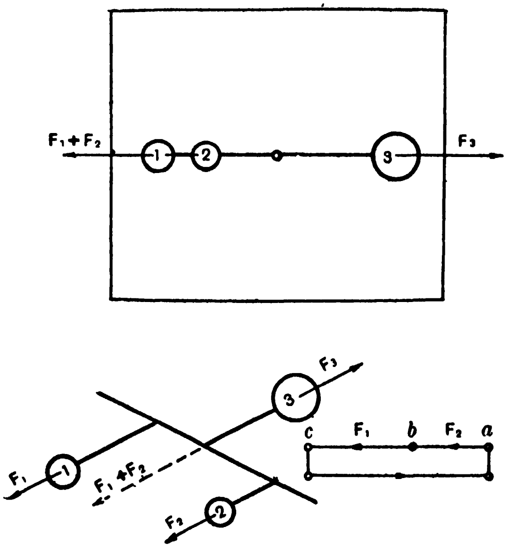


FIG. 92.—Balancing Three Weights Not in the Same Plane.

To make the calculations in an ordinary balancing problem, it is not necessary to know the speed of the shaft, because, although the centrifugal forces to be balanced depend upon the speed of the shaft in number of vibrations per min., this speed is the same for all the weights to be considered, so that an alteration in the speed of the shaft corresponds only to a change in the scale of the vector diagrams.

In our problem the moment of the disturbing couple $= 10 \times 12 \times 9 = 1{,}080$ in.-lb. (it is nearly always more convenient to work in pound and inch units). The centrifugal force due to the balance weights is equivalent to $4 \times 20 = 80$ lb.

$$\therefore \text{ Distance apart of balance weights} = \frac{1{,}080}{80}$$
$$= 13{\cdot}5 \text{ in.}$$

(2) *Three Weights Not in the Same Plane.*—If we have

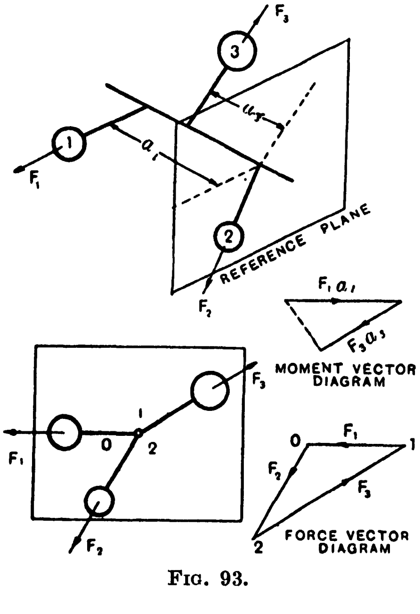


FIG. 93.

three weights rotating about the same axis in different planes, we shall see that they can balance only if their projections on the reference plane are in a straight line, *i.e.* the crank-arms lie all in the same plane. This condition is represented in Fig. 92. It is clear that this system will be balanced if $F_3 = F_1 + F_2$, and $F_3$ is placed at a point along the axis of rotation at which the resultant of $F_1$ and $F_2$ acts.

Suppose that the crank-arms do not lie in the same plane, but are as shown in Fig. 93. It is then quite easy to choose

the weights so that the force vector diagram closes, *i.e.* forms a triangle 0, 1, 2; but if we take our reference plane through one of the weights, say 2, it is clear that the force $F_2$ will have zero moment about this plane, and that the moment vector diagram can close only if 0, 1 and 1, 2 are in the same straight line. This means that the forces $F_1$, $F_3$ must also act in the same straight line, and that, for the force vector diagram to close under this condition, the force $F_2$ must also act in this straight line.

(3) *Four Weights Rotating in Different Planes.*—If we have four weights rotating in different planes about a common axis, there are an indefinite number of ways in which they can be arranged so as to balance.

Since the force vector and moment vector diagrams have both to close for a condition of balance, and this will occur or not irrespective of what scale we choose for our diagrams, it is clear that we need consider only the ratios of the forces and moments; we may also, for simplification, take the plane of rotation of one of the weights as the reference plane, and consider the crank-angles from one of the given weights. We have in the most general examples of our case four forces, at four angles and at four distances from a reference plane; by the simplification outlined above these may be reduced to variable forces at three variable angles at three distances, thus leaving us with nine variable quantities which we have to choose so that they will satisfy our two conditions of balance.

If we consider three of the forces and angles fixed, the closing line of the force vector diagram will determine two more of the variable quantities, *i.e.* the fourth force and the fourth angle; and if we consider two of the arms $a$ as fixed, we are left with the two remaining arms to choose so that the moment vector diagram may close.

This will be made clear from a consideration of Fig. 94; we assume that the force $F_1$ is of unit value, and that the forces $F_2$, $F_3$, arm $a_2$, and angles $\theta_2$, $\theta_3$ are given any values.

On then drawing the force vector diagram we obtain the closing line 3, 0, which gives us the force $F_4$ and angle $\theta_4$, which fix two more variable quantities. Taking the reference plane as that containing the force $F_1$, we see that the moment of $F_1$ about the reference plane is zero, so that there will be no corresponding vector on the moment vector diagram. We therefore start the moment vector diagram by drawing 0, 1 parallel to $F_2$ to represent $F_2 a_2$; then draw from 0 a line 0, 2 parallel to $F_4$, and through 1 a line 1, 2 parallel to $F_3$, the intersection 2 determining the length of the vectors for the moment vector diagram to close, and thus determining the remaining two arms $a_3$, $a_4$. Out of the nine variable quantities, therefore, to which we can reduce those arising in the problem, five can be chosen at random, and the remaining four must be determined from a consideration of the two conditions of balance.

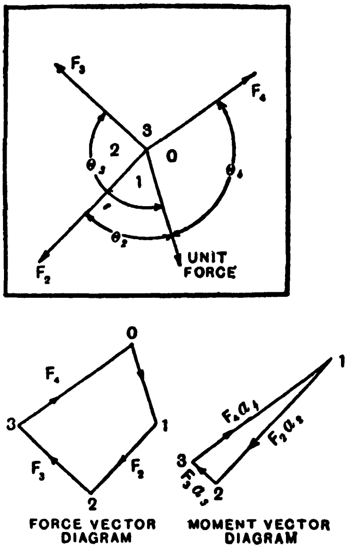


FIG. 94.—Balancing Four Weights Rotating in Different Planes.

In a similar way we can show that if there are $n$ weights, the number of variables can be reduced to $3(n-1)$, and the number that we may fix at random is $3(n-1)-4$, *i.e.* $3n-7$.

**Schubert's Construction for Four Rotating Weights.**—The following graphical construction enables us in an interesting manner to determine crank-angles which will lead to balance when the planes of revolution are given, or will enable the planes of revolution to be determined when the crank-angles are known.

Let 1, 2, 3, 4, Fig. 95, represent four planes of revolution, the centrifugal forces due to the rotation of the corresponding weights being $F_1$, $F_2$, $F_3$, and $F_4$.

Take any convenient point X and join it to the points 1, 2, 3, 4. On X 4 set out to convenient scale the first force $F_1$ and draw *a c* parallel to the axis of rotation 1, 4, cutting

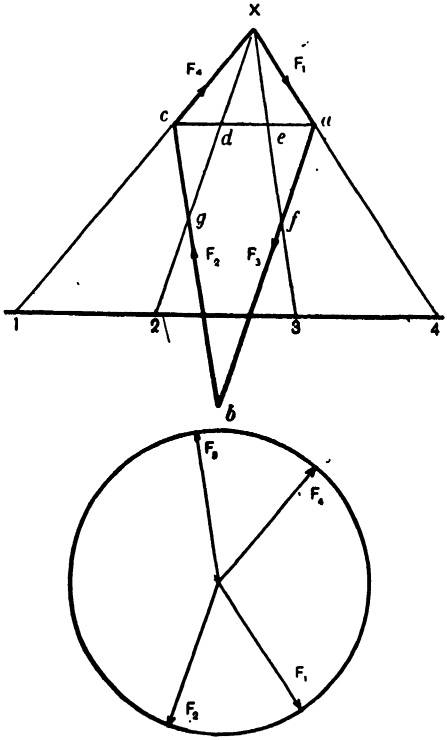


FIG. 95.—Balancing: Schubert's Construction.

X 2 in *d* and X 3 in *e*; next draw *a b* parallel to X 2, and *c b* parallel to X 3, the intersection giving the point *b*.

Then X *a b c* will represent the force vector diagram for a condition of balance so that *a b* gives $F_3$, *b c* gives $F_2$, and *c* X gives $F_4$; and if we draw radial lines parallel to these vectors, we obtain the necessary crank-angles.

We can prove this construction in the following manner: We will take the given forces $F_1$, $F_2$, $F_3$, and $F_4$, which form a closed polygon, and therefore fulfil the first condition of balance, and prove that they fulfil also the second condition.

Consider the triangle $X a f$; $X a$ is parallel to $F_1$, $a f$ to $F_3$, and $f X$ to $F_2$. Now, since $e f$ is parallel to $b c$ and $a c$ to 1, 4, we have

$$\frac{a f}{a b} = \frac{a e}{a c} = \frac{3,\,4}{1,\,4} = \frac{a_3}{a_1};$$

but $a b = F_3$.

$$\therefore\ a f = \frac{F_3 a_3}{a_1} \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

Again, since $X f$ is parallel to $b c$, $X g$ is parallel to $a b$, and $a c$ is parallel to 1, 4, we have

$$\frac{f X}{b c} = \frac{a d}{a c} = \frac{2,\,4}{1,\,4} = \frac{a_2}{a_1};$$

but $b c = F_2$.

$$\therefore\ f X = \frac{F_2 a_2}{a_1}; \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

also

$$X a = F_1 = \frac{F_1 a_1}{a_1} \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

We see, therefore, that, taking the reference plane through the point 4, the force $F_4$ has a zero moment, and the sides of the triangle $X a f$ are proportional to the moments of the centrifugal forces about the point 4, so that this triangle is a moment vector diagram, and, being closed, fulfils the second condition of balance.

Similarly, we could prove that the triangle $X c g$ is a moment vector diagram for a reference plane through the point 1.

We have proved, therefore, that if we choose our revolving weights and radii so that their consequent centrifugal forces are given by the polygon $X a b c$, and their angular relation is that shown, the system will balance.

NUMERICAL EXAMPLE.—We will conclude our treatment with the following numerical example :

*Find balance weights for an inside cylinder engine, with cranks at right angles, the right-hand crank leading. The stroke is* 26 *in., and the equivalent weight to be balanced for each crank is* 1,011 *lb. Distance centre to centre of cylinders* = 23 *in. ; distance between planes containing balance weights* = 59 *in.*

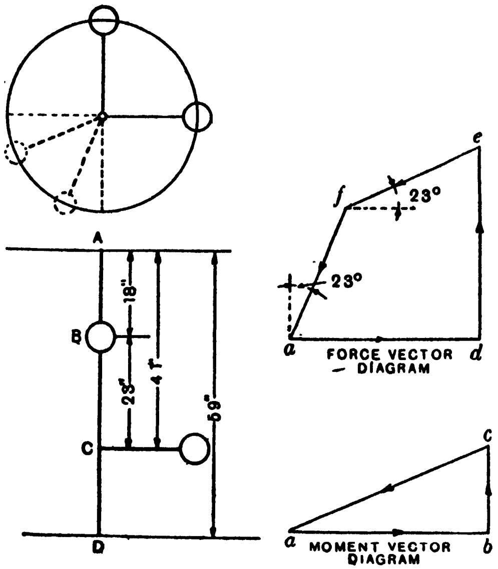


FIG. 96.—Balancing: Numerical Example.

Taking the crank radius of 13 in. as unity, and taking our reference plane through the point A, we shall have $F_1 = F_2 = 1{,}011$, and

$$F_1 a_1 = 18 \times 1{,}011 = 18{,}198 = \text{say, } 18{,}200,$$
$$F_2 a_2 = 41 \times 1{,}011 = 41{,}451 = \text{say, } 41{,}500.$$

Therefore, if we draw *a b*, Fig. 96, to represent 41,500, and *b c* to represent 18,200, the closing line *c a* of the moment vector diagram will give the moment about A of the centrifugal

force due to the balance weight $W_D$ placed at D. By scaling we find $ca = W_D a_3 = 45{,}200$.

$$\therefore\ W_D = \frac{45{,}200}{59} = 766 \text{ lb.}$$

Also the angle $\theta$, which gives the direction of the balance weight with $F_2$ produced, comes 23°. If we draw the force vector polygon $adef$, we shall find that the closing line $fa$, which gives the balance weight at A, comes also equal to 766 lb., and is at 23° to the direction of $F_1$ produced.

If we wish to place the balance weights at radii $r_D$, $r_A$ respectively, we must have

$$W_D = \frac{766 \times 13}{r_D},$$

and

$$W_A = \frac{766 \times 13}{r_A}.$$

The angular positions will still be at 23°, as shown.

# EXERCISES

## CHAPTER I

### THE SUM CURVE AND ITS APPLICATIONS TO WORK CURVES

1. Find by graphical construction the area of a semicircular sheet of 5 in. radius. *Ans.* 39·3 *sq. in.*

2. A load weighing 1,200 lb. is raised by means of a rope provided with an arrangement for indicating the pull at any instant. If the following readings were obtained, find approximately the work done on the load.

| Height above ground (ft.)... | 0 | 10 | 20 | 35 | 50 | 65 | 80 |
|---|---|---|---|---|---|---|---|
| Pull in rope (lb.) ... ... | 2,000 | 1,950 | 1,880 | 1,800 | 1,750 | 1,650 | 1,500 |

*Ans.* 142,000 *ft.-lb.*

3. A tramcar weighs 12·88 tons and starts from rest, the resistance of the rails being constant and equal to 500 lb. Draw the curve of work spent in increasing K.E. against the space described if the tractive forces are as given below.

| Space (ft.) .. ... ... | 0 | 20 | 50 | 80 | 110 | 130 | 160 | 190 | 200 |
|---|---|---|---|---|---|---|---|---|---|
| Force (lb.) ... ... ... | 1,280 | 1,270 | 1,220 | 1,110 | 905 | 800 | 720 | 670 | 660 |

*Ans. Total work at end*$=$95,000 *ft.-lb. approx.*

4. A body weighing 1,610 lb. was lifted vertically by a rope, there being a damped spring balance to indicate the pulling force F lb. of the rope. When the body had been lifted $x$ ft. from its position of rest, the pulling force was automatically recorded as follows :

| $x$ | 0 | 11 | 20 | 34 | 45 | 55 | 66 | 76 |
|---|---|---|---|---|---|---|---|---|
| F | 4,010 | 3,915 | 3,763 | 3,532 | 3,366 | 3,208 | 3,100 | 3,007 |

Find approximately the work done on the body when it has risen 70 ft. How much of this is stored as potential energy, and how much as kinetic energy ? *Ans.* 247,000 ; 112,700 ; 134,300 *ft.-lb.*

5. Each of the two cylinders in a locomotive engine is 16 in. in diameter, and the length of the crank is 12 in. If the driving wheels make 105 revolutions per min., and the mean effective steam pressure is 85 lb. per sq. in., what is the I.H.P. ? *Ans.* 218 *H.P.*

6. What must be the effective H.P. of a locomotive which moves at the steady speed of 35 miles an hour on level rails, the weight of the engine and train being 120 tons, and the resistances 16 lb. per ton? What additional H.P. would be necessary if the rails were laid along a gradient of 1 in 112? *Ans.* 179; 224 *H.P.*

7. Four hundredweight of material is drawn from a depth of 80 fathoms by a rope weighing 1·15 lb. per ft. How much work is expended? *Ans.* 347,000 *ft.-lb.*

8. Find the work done when 10 cu. ft. of air at an initial absolute pressure of 45 lb. per sq. in. is expanded to a volume of 50 cu. ft., the expansion curve being a hyperbola. *Ans.* 104,300 *ft.-lb.*

9. The two cylinders of a locomotive are each 17 in. in diameter, the stroke being 24 in. The mean pressure on the pistons is 80 lb. per sq. in., and the diameter of the driving wheels is 6 ft. Find the horse-power exerted by the engine when the train runs at 30 miles per hour. *Ans.* 616.

10. A pump is worked directly from the ram of a water-pressure engine, the cylinder of which is 6 in. in diameter, that of the pump being $8\frac{3}{4}$ in. The head of water in the supply pipe which gives the pressure is 450 ft., and that in the delivery pipe 150 ft. Find the ratio of work done to total work expended. *Ans.* ·708 : 1.

## CHAPTER II

### SPACE, VELOCITY, AND ACCELERATION CURVES

1. A train which has constant acceleration starts from rest, and at the end of 3 secs. has a velocity with which it would travel through 1 mile in 5 mins. What is the acceleration? *Ans.* 5·87 *ft. per sec.*$^2$.

2. $x$ and $t$ are the distances in miles and the time in hours of a train from a railway terminus.

| $x$ | 0 | 1·5 | 6·0 | 14·0 | 19·0 | 21·0 | 21·5 | 21·8 | 23·0 | 24·7 | 26·8 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | 0 | ·1 | ·2 | ·3 | ·4 | ·5 | ·6 | ·7 | ·8 | ·9 | 1·0 |

What is the greatest speed, and where approximately does it occur?
*Ans.* 80 *miles per hour after* ·2 *hour.*

3. A motor-car starting from rest has the following velocities at the given times. Plot a curve showing the distance covered up to 50 secs., and find the acceleration at 12 and 40 secs.

| Time (secs.) ... ... | 0 | 4 | 9 | 17 | 24 | 30 | 32 | 40 | 45 | 53 |
|---|---|---|---|---|---|---|---|---|---|---|
| Velocity (ft./sec.) .. | 0 | 11·0 | 22·6 | 35·6 | 44·5 | 49·0 | 48·9 | 40·6 | 33·7 | 26·8 |

*Ans.* 1,730 *ft. covered*; $+1·7$ *and* $-1·4$ *ft. per sec.*$^2$.

4. The following numbers give the speed $v$ of a train in miles per hour at times $t$ hours since leaving a station.

| $v$ | 0 | 2·4 | 4·7 | 7·2 | 9·6 | 12·0 | 14·3 | 16·9 | 18·9 | 20·7 | 22·2 | 23·4 | 24·3 | 24·0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | 0 | ·04 | ·08 | ·12 | ·16 | ·20 | ·24 | ·28 | ·32 | ·36 | ·40 | ·44 | ·48 | ·52 |

Draw a diagram showing the distance covered at the various times, and find the total distance covered. *Ans.* 7·56 *miles.*

5. A train goes a distance of 120 miles in 3 hours. During the 1st hour the speed rises uniformly from rest; during the 2nd it remains constant, and during the 3rd it falls uniformly to rest. What is the speed during the 2nd hour? *Ans.* 60 *miles per hour.*

6. A train goes a distance of 5 miles in 8 mins., first moving with constant acceleration and then with equal retardation. Find its greatest speed. *Ans.* 75 *miles per hour.*

7. The speed of a motor-car is determined by observing the times of passing a number of marks placed 500 ft. apart. The time of traversing the distance between the first and second posts was 20 secs., and between the second and third was 19 secs. If the acceleration of the car is constant, find its magnitude and the velocity at the instant of passing the first post. *Ans.* ·67 *ft. per sec. per sec.*; 24·3 *ft. per sec.*

8. In an electric railway the average distance between stations is $\frac{1}{2}$ mile, and running time from start to stop $1\frac{1}{2}$ mins. Between the end of acceleration and the beginning of retardation the speed is a constant one of 25 miles per hour. If the acceleration and retardation are uniform and numerically equal, find their values. If the weight of the train is 150 tons, and the frictional resistance 11 lb. per ton, find the tractive force necessary to start on the level.

*Ans.* 2·04 *ft. per sec. per sec.*; 10·2 *tons.*

9. In a bicycle the length of the cranks is 7 in., the diameter of the back wheel is 28 in., and the gearing is such that the wheel rotates twice as fast as the pedals. If the weight of the machine and cyclist is 160 lb., estimate the force which will have to be applied to the pedals to increase the speed uniformly from 4 to 12 miles an hour in 20 secs., frictional forces being neglected. *Ans.* 14·6 *lb.*

10. A tramcar weighing 15 tons has the power cut off at a certain instant at which the velocity is 16 miles per hour. Measuring times from this instant, the following velocities were noted:

| | | | | |
|---|---|---|---|---|
| Times (seconds) ... ... ... | 0 | 9·3 | 21 | 35 |
| Velocities (miles per hour) ... | 16 | 14 | 12 | 10 |

Calculate the average value of the retarding force and the average velocity during the 35 secs. *Ans.* 264 *lb.*; 12·7 *miles per hour.*

## CHAPTER III

### CURVES OF ACTION AND CONSTRUCTIONS THEREFOR

1. A body weighing 322 lb. is lifted by a force which varies as follows:

| Lift (ft.) .. | 0 | 1 | 2 | 3 | 4 | 5·5 | 7 | 9 | 11 | 12·5 | 14 | 17 onwards |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Force (lb.) | 540 | 540 | 540 | 530 | 500 | 460 | 310 | 220 | 190 | 190 | 190 | 190 |

Find the velocity in each position and determine by graphical construction the time before the body comes to rest. The initial velocity is 5 ft. per sec. *Ans. Stops at* 18·2 *ft.*; 1·92 *secs.*

2. A body weighing 3,220 lb. is lifted by a rope, there being a damped spring balance to indicate the pulling force in the rope; the following values were obtained:

| Lift (ft.) ... ... | 0 | 18 | 43 | 60 | 74 | 95 | 111 | 130 |
|---|---|---|---|---|---|---|---|---|
| Force (lb.) ... ... | 7,700 | 7,680 | 7,430 | 7,130 | 6,770 | 5,960 | 5,160 | 3,970 |

Determine from a curve of velocities plotted against lift the velocity after a lift of 20, 50, 80, 120 ft., and find the time taken to rise from 75 to 85 ft. *Ans.* 34·6; 53·6; 65; 69·5 *ft. per sec.*; ·15 *sec.*

3. A body weighing 15 lb. vibrates simply with a total amplitude of movement of 3 ft. Calculate the forces acting on the body at the ends of the swing and show by a diagram the variation of this force in every position. There are 200 complete vibrations per min. *Ans.* 1,230 *lb.*

4. In Question 3, Chapter I., draw the curve of action and measure the velocity when the space described is 100 ft. *Ans.* 12·3 *ft. per sec.*

5. A body increases uniformly in speed from rest, the speed being 6 ft. per sec. after 4 secs. Find graphically the curve of action. *Ans. Velocity after* 9 *ft.* = 5·2 *ft. per sec.*

6. A locomotive running between two stations 1·6 miles apart gives the train starting from one station an acceleration of 25 miles per hour per half min. until the speed becomes 30 miles per hour. This speed is maintained until the brakes are applied, giving a constant retardation of 3 ft. per sec. per sec., which brings the train to rest at the second station. What is the time taken on the journey? *Ans.* 3·6 *mins.*

7. A particle moves with simple harmonic motion, the stroke being 10 ft., and the time of a complete movement 4 secs. Find the time taken in passing between points 4 ft. and 2 ft. from the centre of the motion and on the same side of the centre. *Ans.* ·33 *sec.*

8. A car weighs 12 tons, and the tractive force F lb. at distances $x$ ft. from the initial position was noted as follows :

| $x$ | 0 | 10 | 30 | 50 | 65 | 80 | 94 | 100 |
|---|---|---|---|---|---|---|---|---|
| F | 1,440 | 1,390 | 1,250 | 1,060 | 910 | 805 | 760 | 740 |

If the constant resistance of the road is equivalent to 600 lb., draw a curve of velocities plotted against distances from the starting point.

*Ans. Velocity at end* = 10·5 *ft. per sec.*

## CHAPTER IV

### POLAR DIAGRAMS

1. Draw the polar displacement and velocity diagrams for the case given in Question 2 of Exercises II.

2. Solve Question 3 of Exercises II. by means of polar diagrams.

3. A simple slide valve has a travel of 5 in. and may be considered to have simple harmonic motion. The cut-off is at $\frac{5}{8}$ of the stroke of the piston, and the release takes place at $\frac{7}{8}$ of the stroke. The lead is $\frac{1}{8}$ in. Assuming that the piston has also simple harmonic motion, find the outside and inside laps and the compression point.

*Ans.* 1·46 *in.* ; ·1 *in.* ; ·89 *of the stroke.*

4. The slide valve of a horizontal engine is driven from a point C in a link A B of such length that its angular deviation from the vertical is small. The horizontal motions of A and B are simple harmonic defined as follows : For A half travel = 2·5 in., advance = 20° ; for B half travel = 2·5 in., advance = 160°. If A C = $\frac{1}{4}$ A B, find the half travel and advance of C. *Ans.* 1·46 *in.* ; 36·5°.

5. Draw the profile of a cam to do the following work : It has to lift a bar vertically with uniform velocity, the length of travel of the bar being 6 in. ; it then has to allow the bar to descend again with uniform velocity, but at one-half the speed of the ascent. The two movements occupy one revolution of the uniformly rotating cam. The diameter of the roller working on the cam is $\frac{1}{2}$ in., and the least thickness of metal round the cam centre must be 2 in. The line of stroke of the moving bar passes through the cam centre.

## CHAPTER V

### DIAGRAMS FOR VELOCITY CHANGING IN DIRECTION ; THE HODOGRAPH

1. A train is travelling at 20 miles an hour, and a man, sitting in a compartment with both windows open, observes a stone pass through both windows at right angles to the direction of the train. If the stone appears to move 20 ft. per sec. to the man, with what velocity was it thrown ? *Ans.* 35·5 *ft. per sec.*

2. A cyclist is riding due W. at 12 miles per hour, and the wind is blowing from the S.E. at 5·5 miles per hour. If the cyclist carries a small flag, in what direction will it fly? At what speed would the cyclist have to ride to make the flag fly due N.?

*Ans.* 25° 37′ *N.E.*; 3·89 *miles per hour.*

3. If a steamer is going due west at 14 knots, and the wind blows from the north with a velocity of 7 knots, what will be the apparent direction of the wind to a passenger on the steamer? *Ans.* 63° *W. of N.*

4. A link A B, 2 ft. long, is horizontal at a given instant, and is moving in a vertical plane. The velocity of the left-hand end A is 10 ft. per sec. upwards to the right at 45° to A B. The velocity of B relative to A is 4 ft. per sec. downwards. What is the actual velocity of B?

*Ans.* 7·7 *ft. per sec. upwards to the right at* 23½° *to A B produced.*

5. At midnight a certain vessel A was 40 miles due N. of a vessel B, A steaming a course S.W. at 20 miles per hour, and B due W. at 12 miles per hour. When will they be 10 miles apart, and when will they again become farther apart? *Ans.* 2.11 *a.m.*; 3.7 *a.m.*

6. A rope brake is fitted to a fly-wheel, 3 ft. in diameter to the centre of the rope and running at 220 revolutions per min. What must be the difference in pull at the two ends of the rope to transmit 7 B.H.P.?

*Ans.* 111 *lb.*

7. A jet of water 1 in. in diameter falling from a height of 200 ft. strikes a fixed hemispherical cup so as to reverse its direction. Find the force which it exerts on the cup, assuming that the jet has 90% of the full velocity due to its height of fall. *Ans.* 221 *lb.*

8. A locomotive engine weighs 38 tons and travels round a curve of 800 ft. radius at 50 miles an hour. Find the centrifugal force and the direction and magnitude of the resultant thrust on the rails due to its weight and the centrifugal force.

*Ans.* 8 *tons approx.*; 38·8 *tons at* 12° *to vertical.*

9. A brake wheel, 4 ft. in diameter, on a horizontal axle, is furnished with internal flanges which, along with the rim, form a trough containing cooling water. What is the least speed which will prevent the water from falling out? *Ans.* 38·2 *revolutions per min.*

10. In an inward-flow turbine the inner and outer diameters of the wheel are 1 and 2 ft. respectively. The water enters the outer circumference at 12° to the tangent and leaves the inner circumference radially. The radial velocity of flow in both cases is 6 ft. per sec., and the wheel makes 3·6 revolutions per sec. Determine the angles of vanes at inlet and outlet. *Ans.* 47·5°; 27·3°.

11. An inward-flow turbine works under a head of 60 ft. at 380 revolutions per min.; diameter of outer circumference 24 in., inner ditto 12 in. Water enters at 44 ft. per sec. at an angle of 10° to the tangent. Assuming the velocity of flow through the wheel to be constant, determine the direction of the tangent to the vane at inlet and outlet and the hydraulic efficiency of the turbine.

*Ans.* 63·5° *outlet*; 21° *inlet*; 89·7%.

## CHAPTERS VI AND VII

### VELOCITIES, ACCELERATIONS, AND STATIC FORCES IN MECHANISMS

1. Construct a curve to show the velocity of the cross-head at all points of the stroke for a uniformly revolving crank of 1 ft. radius, the length of the connecting-rod being 3 ft. From this curve construct an acceleration curve, and check this curve by finding the acceleration at a number of points by direct graphical construction.

2. In a crank and slotted-lever quick-return motion the stroke is 8 in., and the ratio of times of home and cutting strokes is 3 : 5. The line of the stroke of the ram produced passes through the extreme positions of the connecting-rod pin at the end of the slotted lever. If the distance between the centre of the driving plate and the axis about which the slotted lever oscillates is 6 in., find the crank radius and length of the lever. *Ans. Radius* = 2·29 *in.* ; *length of lever* 10·46 *in.*

3. In a four-bar chain (Fig. 58) the crank-shafts are 2·5 ft. apart; the long and short cranks are 1·65 and 1·2 ft. long respectively, and the connecting link is 1·25 ft. long. The end of the short crank rotates with a uniform peripheral speed of 10 ft. per sec. What is the velocity of the end of the other crank when the first makes 60° with the line joining the shafts? *Ans.* 6·25 *ft. per sec.*

4. In a four-bar chain (Fig. 58) the length C D = 8 in.; C B = B A = 4·5 in.; A D = 4 in. Midway between B and A is an arm integral with this link which projects upwards at 90° to the link, the length of the arm being $1\frac{1}{2}$ in., the top point being called E. Find the velocities of B and E for equal angular displacements of A D of 30° between horizontal and vertical positions if A moves with a uniform tangential velocity of 10 ft. per sec.; and if a uniform tangential force of 10 lb. acts on A D at A, find the tangential and horizontal resistances at E and B which can be overcome.

*Ans.*

| Angle A D C | Velocity of E | Velocity of B | Tangential Force (lb.) | | Horizontal Force (lb.) | |
|---|---|---|---|---|---|---|
| | | | at E | at B | at E | at B |
| 90° | 21 ft./sec. | 39·5 ft./sec. | 4·76 | 2·53 | 21·0 | 4·1 |
| 60° | 5·9 ,, | 7·8 ,, | 16·9 | 12·8 | 18·3 | 13·6 |
| 30° | 5·8 ,, | 0·6 ,, | 17·3 | 1·67 | 16·6 | 10·6 |
| 0° | 3·2 ,, | 11·3 ,, | 2·58 | 0·85 | 14·9 | 9·8 |

5. In the oscillating cylinder mechanism the point B (see Fig. 65) is fixed, and the connecting-rod C B carries a slide at B which oscillates about the point. If the crank radius is 18 in., the length O B 5 ft., and the crank makes 60 revolutions per min., find the maximum angular velocity of the oscillating slide (cylinder) and draw polar angular velocity curves. *Ans. Max. angular velocity* = 2·69 *radians per sec.*

6. The crank in a steam-engine mechanism is 8 in. long, and the

length of the connecting-rod is 36 in. If the crank-shaft makes 200 revolutions per min., find the acceleration of the piston when the crank is 30° from the inner dead centre. *Ans.* 286 *ft. per sec. per sec.*

7. Prove from graphical considerations that in the steam-engine mechanism in which the connecting-rod is $k$ cranks long, and the crank is at right angles to the connecting-rod, the velocity of the piston rod is $\frac{2\pi n r\sqrt{1+k^2}}{k}$, and the acceleration is $\frac{4\pi^2 n^2 r\sqrt{1+k^2}}{k^4}$, $r$ being the crank radius and $n$ the revolutions per sec.

8. In a four-bar chain (Fig. 58) D A = 3 ft.; A B = 2·25 ft.; B C = 3·9 ft.; and C D = 4·63 ft. If the angle A D C = 60°, and the shaft D rotates at 120 revolutions per min., find the angular acceleration of the other shaft. *Ans.* 198 *radians per sec.*

9. If it is essential to prevent reversal of thrust in a steam-engine running at 60 revolutions per min., to what pressure should compression be carried in the case of an engine of stroke 4 ft., length of connecting-rod 9 ft., and weight of reciprocating parts per sq. in. of piston 3·2 lb.? *Ans.* 9·57 *and* 6·09 *lb. per sq. in. at "in" and "out" ends.*

10. In a crank and connecting-rod mechanism the length O C is 2·5 ft., O B is 3·6 ft., B C is 1·5 ft., and the connecting-rod B C is extended beyond C to a point P, the length C P being ·8 ft. A force of 2,000 lb. acts at B towards O. What weight hung vertically at P will balance this force? *Ans.* 526 *lb.*

11. In a quick-return mechanism (Fig. 61), the crank is 5 in. long, and the distance O P is 15 in., the length P B being 25 in. When the crank rotates at 100 revolutions per min., what are the highest and lowest velocities of the point B?

*Ans.* 10·9 *and* 5·45 *ft. per sec.*

## CHAPTER VIII

### ELEMENTS OF FLY-WHEEL DESIGN

1. An iron disc weighing 450 lb. per cu. ft. is 8 in. in diameter, 2 in. thick, and runs at 2,500 revolutions per min. What will be the percentage increase in speed if it absorbs an additional 200 ft.-lb. of energy? *Ans.* ·0035.

2. A fly-wheel 5 ft. 3 in. in diameter has a rim weighing 1,000 lb. Find the number of ft.-lb. of work required to set the fly-wheel rotating at 120 revolutions per min. *Ans.* 17,000 *ft.-lb.*

3. A certain fly-wheel gives out 6,500 ft.-lb. of kinetic energy in changing its speed from 170 to 168 revolutions per min. What is the kinetic energy of this wheel when its speed is 172 revolutions per min.? *Ans.* 285,000 *ft.-lb.*

4. An engine of 100 I.H.P. runs at 200 revolutions per min. The

energy to be absorbed by the fly-wheel between its maximum and minimum speeds is 10% of the work done per revolution. If the gyration radius of the fly-wheel is 2 ft. 6 in., find its weight in order that the total fluctuation in speed may not exceed 2 per cent. of the mean speed. *Ans.* 970 *lb.*

5. If the average piston speed of an engine is 400 ft. per min., and $\frac{1}{3}$ of the work per stroke has to be absorbed by the fly-wheel with a fluctuation of velocity of 1% on either side of the mean, the radius of the fly-wheel being twice the stroke, show that the weight of the rim may be taken as 50 lb. per I.H.P. for a single-cylinder engine running at 100 revolutions per min.

6. The fly-wheel of an engine of 4 H.P. running at 75 revolutions per min. is equivalent to a heavy rim 2 ft. 9 in. mean diameter and weighing 500 lb. Find the maximum and minimum speeds of rotation when the fluctuation of energy is one-fourth of the energy of one revolution.

*Ans.* 84 ; 66 *revolutions per min.*

7. Determine the weight of rim per horse-power of an engine which, when running at a speed of 70 ft. per sec., will have stored in it 10% of the energy developed per min. *Ans.* 43·1 *lb.*

8. In a 25-H.P. engine the fluctuation of energy may be taken to be $\frac{1}{4}$ of the work done in the cylinders per revolution, and the maximum and minimum revolutions of the crank-shaft per min. have to be 131 and 129. If the diameter of the fly-wheel is 5 ft., estimate the weight of metal that must be concentrated in the rim, the boss and arms being neglected. *Ans.* 2,870 *lb.*

## CHAPTER IX

### BALANCING ROTATING PARTS

1. A fly-wheel weighing 5 tons has its centre of gravity $\frac{1}{10}$ of an in. from the centre of the shaft. Find the force upon the shaft caused by the lack of balance when running at 200 revolutions per min.

*Ans.* ·57 *ton.*

2. Two weights of 10 and 20 lb. respectively are attached to a balanced disc 90° apart and at radii 2 ft. and 3 ft. respectively. Find the resultant force on the axis at 200 revolutions per min., and find the angular position and magnitude of a weight placed at $2\frac{1}{2}$ ft. radius which will balance at all speeds.

*Ans.* 8,620 *lb.* ; 25·3 *lb. at* 108·4° *to the* 10 *lb. weight.*

3. Find the weight of balance weights at 33 in. radius and 60 in. apart which will balance reciprocating and rotating parts weighing 500 and 680 lb. respectively of a two-cylinder engine with crank radius 12 in. and distance apart of cylinders 24 in. *Ans.* 325 *lb.*

4. A shaft runs in bearings A, B, 15 ft. apart, and carries three pulleys C, D, E, weighing 360, 400, and 200 lb. respectively, and are placed at 4, 9, and 12 ft. from A. Their centres of gravity are distant

from the shaft centre line by amounts C, $\frac{3}{16}$ in.; D, $\frac{1}{8}$ in.; and E, $\frac{1}{4}$ in. Arrange the angular positions of the pulleys so that there shall be no dynamic force at B, and find for that arrangement the dynamic force at A when the shaft runs at 100 revolutions per min.

*Ans.* $C = 166°$, $D = 235°$, $E = 30°$; 14·7 *lb.*

5. Find the positions and magnitudes of the balance weights required to balance all the revolving and $\frac{2}{3}$ of the reciprocating masses in a simple inside cylinder locomotive specified as follows: Masses per cylinder at 12 in. radius, 720 lb. revolving, 630 lb. reciprocating; cylinders, 26 in. centre line to centre line; planes of balance weights, 58 in. apart; radius of balance weights, 32 in.

*Ans.* 332 *lb. at* 159° *to adjacent crank.*

6. The reciprocating masses for the 1st, 2nd, and 3rd cylinders in a four-cylinder engine weigh 4, 6, and 8 tons, and the centre lines of these cylinders are respectively 13, 9, and 4 ft. from that of the 4th cylinder. Find the 4th reciprocating mass and the angles between cranks for them to be balanced.

*Ans.* 5·15 *tons; angles as follows with* 1*st crank: No.* 2, 145°; *No.* 3, 256°; *No.* 4, 55°.

7. The weight of crank arms and pin of a crank-shaft are equivalent to a weight of 700 lb. at 12 in. radius. The shaft is supported on two bearings 5 ft. apart, the centre of the crank being 18 in. from the left-hand bearing. Find the dynamic load on the shaft and bearings for a speed of 240 revolutions per min., and find the maximum bending moment on the crank-shaft.

*Ans.* 13,700 *lb. on shaft;* 9,600 *lb. on left-hand bearing;* 4,100 *lb. on right; B.M.* = 173,000 *in.-lb.*

8. The reciprocating masses in a four-crank engine weigh respectively $5\frac{1}{2}$, 7, 6, 5 tons taken in order. Find the cylinder centre lines, having given that the pitch of the extreme cylinders is 39 ft., and that the remaining two are to be arranged symmetrically with respect to them. Find also the corresponding set of crank angles.

*Ans. Pitch inner cylinders* 1,575 *ft.; angle between* 1 *and* 3 = 118°; *between* 3 *and* 2 = $81\frac{1}{2}$°; *between* 2 *and* 4 = 123°; *between* 4 *and* 1 = 37·5°.

| Angle | | Chord. | Sine. | Tangent. | Co-tangent. | Cosine. | | | |
|---|---|---|---|---|---|---|---|---|---|
| Degrees | Radians. | | | | | | | | |
| 0° | 0 | 0 | 0 | 0 | ∞ | 1 | 1·414 | 1·5708 | 90° |
| 1 | ·0175 | ·017 | ·0175 | ·0175 | 57·2900 | ·9998 | 1·402 | 1·5533 | 89 |
| 2 | ·0349 | ·035 | ·0349 | ·0349 | 28·6363 | ·9994 | 1·389 | 1·5359 | 88 |
| 3 | ·0524 | ·052 | ·0523 | ·0524 | 19·0811 | ·9986 | 1·377 | 1·5184 | 87 |
| 4 | ·0698 | ·070 | ·0698 | ·0699 | 14·3007 | ·9976 | 1·364 | 1·5010 | 86 |
| 5 | ·0873 | ·087 | ·0872 | ·0875 | 11·4301 | ·9962 | 1·351 | 1·4835 | 85 |
| 6 | ·1047 | ·105 | ·1045 | ·1051 | 9·5144 | ·9945 | 1·338 | 1·4661 | 84 |
| 7 | ·1222 | ·122 | ·1219 | ·1228 | 8·1443 | ·9925 | 1·325 | 1·4486 | 83 |
| 8 | ·1396 | ·140 | ·1392 | ·1405 | 7·1154 | ·9903 | 1·312 | 1·4312 | 82 |
| 9 | ·1571 | ·157 | ·1564 | ·1584 | 6·3138 | ·9877 | 1·299 | 1·4137 | 81 |
| 10 | ·1745 | ·174 | ·1736 | ·1763 | 5·6713 | ·9848 | 1·286 | 1·3963 | 80 |
| 11 | ·1920 | ·192 | ·1908 | ·1944 | 5·1446 | ·9816 | 1·272 | 1·3788 | 79 |
| 12 | ·2094 | ·209 | ·2079 | ·2126 | 4·7046 | ·9781 | 1·259 | 1·3614 | 78 |
| 13 | ·2269 | ·226 | ·2250 | ·2309 | 4·3315 | ·9744 | 1·245 | 1·3439 | 77 |
| 14 | ·2443 | ·244 | ·2419 | ·2493 | 4·0108 | ·9703 | 1·231 | 1·3265 | 76 |
| 15 | ·2618 | ·261 | ·2588 | ·2679 | 3·7321 | ·9659 | 1·218 | 1·3090 | 75 |
| 16 | ·2793 | ·278 | ·2756 | ·2867 | 3·4874 | ·9613 | 1·204 | 1·2915 | 74 |
| 17 | ·2967 | ·296 | ·2924 | ·3057 | 3·2709 | ·9563 | 1·190 | 1·2741 | 73 |
| 18 | ·3142 | ·313 | ·3090 | ·3249 | 3·0777 | ·9511 | 1·176 | 1·2566 | 72 |
| 19 | ·3316 | ·330 | ·3256 | ·3443 | 2·9042 | ·9455 | 1·161 | 1·2392 | 71 |
| 20 | ·3491 | ·347 | ·3420 | ·3640 | 2·7475 | ·9397 | 1·147 | 1·2217 | 70 |
| 21 | ·3665 | ·364 | ·3584 | ·3839 | 2·6051 | ·9336 | 1·133 | 1·2043 | 69 |
| 22 | ·3840 | ·382 | ·3746 | ·4040 | 2·4751 | ·9272 | 1·118 | 1·1868 | 68 |
| 23 | ·4014 | ·399 | ·3907 | ·4245 | 2·3559 | ·9205 | 1·104 | 1·1694 | 67 |
| 24 | ·4189 | ·416 | ·4067 | ·4452 | 2·2460 | ·9135 | 1·089 | 1·1519 | 66 |
| 25 | ·4363 | ·433 | ·4226 | ·4663 | 2·1445 | ·9063 | 1·075 | 1·1345 | 65 |
| 26 | ·4538 | ·450 | ·4384 | ·4877 | 2·0503 | ·8988 | 1·060 | 1·1170 | 64 |
| 27 | ·4712 | ·467 | ·4540 | ·5095 | 1·9626 | ·8910 | 1·045 | 1·0996 | 63 |
| 28 | ·4887 | ·484 | ·4695 | ·5317 | 1·8807 | ·8829 | 1·030 | 1·0821 | 62 |
| 29 | ·5061 | ·501 | ·4848 | ·5543 | 1·8040 | ·8746 | 1·015 | 1·0647 | 61 |
| 30 | ·5236 | ·518 | ·5000 | ·5774 | 1·7321 | ·8660 | 1·000 | 1·0472 | 60 |
| 31 | ·5411 | ·534 | ·5150 | ·6009 | 1·6643 | ·8572 | ·985 | 1·0297 | 59 |
| 32 | ·5585 | ·551 | ·5299 | ·6249 | 1·6003 | ·8480 | ·970 | 1·0123 | 58 |
| 33 | ·5760 | ·568 | ·5446 | ·6494 | 1·5399 | ·8387 | ·954 | ·9948 | 57 |
| 34 | ·5934 | ·585 | ·5592 | ·6745 | 1·4826 | ·8290 | ·939 | ·9774 | 56 |
| 35 | ·6109 | ·601 | ·5736 | ·7002 | 1·4281 | ·8192 | ·923 | ·9599 | 55 |
| 36 | ·6283 | ·618 | ·5878 | ·7265 | 1·3764 | ·8090 | ·908 | ·9425 | 54 |
| 37 | ·6458 | ·635 | ·6018 | ·7536 | 1·3270 | ·7986 | ·892 | ·9250 | 53 |
| 38 | ·6632 | ·651 | ·6157 | ·7813 | 1·2799 | ·7880 | ·877 | ·9076 | 52 |
| 39 | ·6807 | ·668 | ·6293 | ·8098 | 1·2349 | ·7771 | ·861 | ·8901 | 51 |
| 40 | ·6981 | ·684 | ·6428 | ·8391 | 1·1918 | ·7660 | ·845 | ·8727 | 50 |
| 41 | ·7156 | ·700 | ·6561 | ·8693 | 1·1504 | ·7547 | ·829 | ·8552 | 49 |
| 42 | ·7330 | ·717 | ·6691 | ·9004 | 1·1106 | ·7431 | ·813 | ·8378 | 48 |
| 43 | ·7505 | ·733 | ·6820 | ·9325 | 1·0724 | ·7314 | ·797 | ·8203 | 47 |
| 44 | ·7679 | ·749 | ·6947 | ·9657 | 1·0355 | ·7193 | ·781 | ·8029 | 46 |
| 45° | ·7854 | ·765 | ·7071 | 1·0000 | 1·0000 | ·7071 | ·765 | ·7854 | 45° |
| | | | Cosine | Co-tangent | Tangent | Sine | Chord | Radians | Degrees |
| | | | | | | | | Angle | |

## Logarithms

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4<br>4 | 9<br>8 | 13<br>12 | 17<br>16 | 21<br>20 | 26<br>24 | 30<br>28 | 34<br>32 | 38<br>37 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4<br>4 | 8<br>7 | 12<br>11 | 15<br>15 | 19<br>19 | 23<br>22 | 27<br>26 | 31<br>30 | 35<br>33 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 | 3<br>3 | 7<br>7 | 11<br>10 | 14<br>14 | 18<br>17 | 21<br>20 | 25<br>24 | 28<br>27 | 32<br>31 |
| 13 | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3<br>3 | 7<br>7 | 10<br>10 | 13<br>12 | 16<br>16 | 20<br>19 | 23<br>22 | 26<br>25 | 30<br>29 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3<br>3 | 6<br>6 | 9<br>9 | 12<br>12 | 15<br>15 | 18<br>17 | 21<br>20 | 24<br>23 | 28<br>26 |
| 15 | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3<br>3 | 6<br>5 | 9<br>8 | 11<br>11 | 14<br>14 | 17<br>16 | 20<br>19 | 23<br>22 | 26<br>25 |
| 16 | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3<br>3 | 5<br>5 | 8<br>8 | 11<br>10 | 14<br>13 | 16<br>15 | 19<br>18 | 22<br>21 | 24<br>23 |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 | 3<br>2 | 5<br>5 | 8<br>7 | 10<br>10 | 13<br>12 | 15<br>15 | 18<br>17 | 20<br>19 | 23<br>22 |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2<br>2 | 5<br>5 | 7<br>7 | 9<br>9 | 12<br>11 | 14<br>14 | 16<br>16 | 19<br>18 | 21<br>21 |
| 19 | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2<br>2 | 4<br>4 | 7<br>6 | 9<br>8 | 11<br>11 | 13<br>13 | 16<br>15 | 18<br>17 | 20<br>19 |
| 20 | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| 25 | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2 | 3 | 5 | 7 | 8 | 10 | 11 | 13 | 15 |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 30 | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 12 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 35 | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 38 | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| 40 | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 |
| 41 | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 43 | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 44 | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 45 | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 46 | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 47 | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 50 | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |

## Logarithms

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 51 | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 52 | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| 53 | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 54 | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 55 | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 59 | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 60 | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 61 | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 65 | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 70 | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 71 | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 73 | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 74 | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 75 | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 76 | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 77 | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 80 | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 81 | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 82 | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 83 | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 84 | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 85 | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 86 | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 87 | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 88 | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 89 | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 90 | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 91 | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 92 | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 93 | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 94 | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 95 | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 96 | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 97 | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 98 | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 99 | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |

## Logarithms

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4<br>4 | 9<br>8 | 13<br>12 | 17<br>16 | 21<br>20 | 26<br>24 | 30<br>28 | 34<br>32 | 38<br>37 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4<br>4 | 8<br>7 | 12<br>11 | 15<br>15 | 19<br>19 | 23<br>22 | 27<br>26 | 31<br>30 | 35<br>33 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 | 3<br>3 | 7<br>7 | 11<br>10 | 14<br>14 | 18<br>17 | 21<br>20 | 25<br>24 | 28<br>27 | 32<br>31 |
| 13 | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3<br>3 | 7<br>7 | 10<br>10 | 13<br>12 | 16<br>16 | 20<br>19 | 23<br>22 | 26<br>25 | 30<br>29 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3<br>3 | 6<br>6 | 9<br>9 | 12<br>12 | 15<br>15 | 18<br>17 | 21<br>20 | 24<br>23 | 28<br>26 |
| 15 | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3<br>3 | 6<br>5 | 9<br>8 | 11<br>11 | 14<br>14 | 17<br>16 | 20<br>19 | 23<br>22 | 26<br>25 |
| 16 | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3<br>3 | 5<br>5 | 8<br>8 | 11<br>10 | 14<br>13 | 16<br>15 | 19<br>18 | 22<br>21 | 24<br>23 |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2430 | 2504 | 2529 | 3<br>2 | 5<br>5 | 8<br>7 | 10<br>10 | 13<br>12 | 15<br>15 | 18<br>17 | 20<br>19 | 23<br>22 |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2<br>2 | 5<br>5 | 7<br>7 | 9<br>9 | 12<br>11 | 14<br>14 | 16<br>16 | 19<br>18 | 21<br>21 |
| 19 | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2<br>2 | 4<br>4 | 7<br>6 | 9<br>8 | 11<br>11 | 13<br>13 | 16<br>15 | 18<br>17 | 20<br>19 |
| 20 | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| 25 | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2 | 3 | 5 | 7 | 8 | 10 | 11 | 13 | 15 |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 30 | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 12 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 35 | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 38 | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| 40 | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 |
| 41 | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 43 | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 44 | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 45 | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 46 | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 47 | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 50 | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |

## Logarithms

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 51 | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 52 | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| 53 | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 54 | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 55 | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 59 | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 60 | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 61 | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 65 | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 70 | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 71 | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 73 | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 74 | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 75 | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 76 | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 77 | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 80 | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 81 | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 82 | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 83 | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 84 | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 85 | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 86 | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 87 | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 88 | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 89 | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 90 | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 91 | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 92 | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 93 | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 94 | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 95 | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 96 | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 97 | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 98 | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 99 | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |

## Antilogarithms

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·00 | 1000 | 1002 | 1005 | 1007 | 1009 | 1012 | 1014 | 1016 | 1019 | 1021 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·01 | 1023 | 1026 | 1028 | 1030 | 1033 | 1035 | 1038 | 1040 | 1042 | 1045 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·02 | 1047 | 1050 | 1052 | 1054 | 1057 | 1059 | 1062 | 1064 | 1067 | 1069 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·03 | 1072 | 1074 | 1076 | 1079 | 1081 | 1084 | 1086 | 1089 | 1091 | 1094 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·04 | 1096 | 1099 | 1102 | 1104 | 1107 | 1109 | 1112 | 1114 | 1117 | 1119 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·05 | 1122 | 1125 | 1127 | 1130 | 1132 | 1135 | 1138 | 1140 | 1143 | 1146 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·06 | 1148 | 1151 | 1153 | 1156 | 1159 | 1161 | 1164 | 1167 | 1169 | 1172 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·07 | 1175 | 1178 | 1180 | 1183 | 1186 | 1189 | 1191 | 1194 | 1197 | 1199 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·08 | 1202 | 1205 | 1208 | 1211 | 1213 | 1216 | 1219 | 1222 | 1225 | 1227 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| ·09 | 1230 | 1233 | 1236 | 1239 | 1242 | 1245 | 1247 | 1250 | 1253 | 1256 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| ·10 | 1259 | 1262 | 1265 | 1268 | 1271 | 1274 | 1276 | 1279 | 1282 | 1285 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| ·11 | 1288 | 1291 | 1294 | 1297 | 1300 | 1303 | 1306 | 1309 | 1312 | 1315 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| ·12 | 1318 | 1321 | 1324 | 1327 | 1330 | 1334 | 1337 | 1340 | 1343 | 1346 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| ·13 | 1349 | 1352 | 1355 | 1358 | 1361 | 1365 | 1368 | 1371 | 1374 | 1377 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·14 | 1380 | 1384 | 1387 | 1390 | 1393 | 1396 | 1400 | 1403 | 1406 | 1409 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·15 | 1413 | 1416 | 1419 | 1422 | 1426 | 1429 | 1432 | 1435 | 1439 | 1442 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·16 | 1445 | 1449 | 1452 | 1455 | 1459 | 1462 | 1466 | 1469 | 1472 | 1476 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·17 | 1479 | 1483 | 1486 | 1489 | 1493 | 1496 | 1500 | 1503 | 1507 | 1510 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·18 | 1514 | 1517 | 1521 | 1524 | 1528 | 1531 | 1535 | 1538 | 1542 | 1545 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·19 | 1549 | 1552 | 1556 | 1560 | 1563 | 1567 | 1570 | 1574 | 1578 | 1581 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| ·20 | 1585 | 1589 | 1592 | 1596 | 1600 | 1603 | 1607 | 1611 | 1614 | 1618 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| ·21 | 1622 | 1626 | 1629 | 1633 | 1637 | 1641 | 1644 | 1648 | 1652 | 1656 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| ·22 | 1660 | 1663 | 1667 | 1671 | 1675 | 1679 | 1683 | 1687 | 1690 | 1694 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| ·23 | 1698 | 1702 | 1706 | 1710 | 1714 | 1718 | 1722 | 1726 | 1730 | 1734 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| ·24 | 1738 | 1742 | 1746 | 1750 | 1754 | 1758 | 1762 | 1766 | 1770 | 1774 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| ·25 | 1778 | 1782 | 1786 | 1791 | 1795 | 1799 | 1803 | 1807 | 1811 | 1816 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| ·26 | 1820 | 1824 | 1828 | 1832 | 1837 | 1841 | 1845 | 1849 | 1854 | 1858 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| ·27 | 1862 | 1866 | 1871 | 1875 | 1879 | 1884 | 1888 | 1892 | 1897 | 1901 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| ·28 | 1905 | 1910 | 1914 | 1919 | 1923 | 1928 | 1932 | 1936 | 1941 | 1945 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·29 | 1950 | 1954 | 1959 | 1963 | 1968 | 1972 | 1977 | 1982 | 1986 | 1991 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·30 | 1995 | 2000 | 2004 | 2009 | 2014 | 2018 | 2023 | 2028 | 2032 | 2037 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·31 | 2042 | 2046 | 2051 | 2056 | 2061 | 2065 | 2070 | 2075 | 2080 | 2084 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·32 | 2089 | 2094 | 2099 | 2104 | 2109 | 2113 | 2118 | 2123 | 2128 | 2133 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·33 | 2138 | 2143 | 2148 | 2153 | 2158 | 2163 | 2168 | 2173 | 2178 | 2183 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·34 | 2188 | 2193 | 2198 | 2203 | 2208 | 2213 | 2218 | 2223 | 2228 | 2234 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·35 | 2239 | 2244 | 2249 | 2254 | 2259 | 2265 | 2270 | 2275 | 2280 | 2286 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·36 | 2291 | 2296 | 2301 | 2307 | 2312 | 2317 | 2323 | 2328 | 2333 | 2339 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·37 | 2344 | 2350 | 2355 | 2360 | 2366 | 2371 | 2377 | 2382 | 2388 | 2393 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·38 | 2399 | 2404 | 2410 | 2415 | 2421 | 2427 | 2432 | 2438 | 2443 | 2449 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·39 | 2455 | 2460 | 2466 | 2472 | 2477 | 2483 | 2489 | 2495 | 2500 | 2506 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| ·40 | 2512 | 2518 | 2523 | 2529 | 2535 | 2541 | 2547 | 2553 | 2559 | 2564 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| ·41 | 2570 | 2576 | 2582 | 2588 | 2594 | 2600 | 2606 | 2612 | 2618 | 2624 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| ·42 | 2630 | 2636 | 2642 | 2649 | 2655 | 2661 | 2667 | 2673 | 2679 | 2685 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| ·43 | 2692 | 2698 | 2704 | 2710 | 2716 | 2723 | 2729 | 2735 | 2742 | 2748 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| ·44 | 2754 | 2761 | 2767 | 2773 | 2780 | 2786 | 2793 | 2799 | 2805 | 2812 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| ·45 | 2818 | 2825 | 2831 | 2838 | 2844 | 2851 | 2858 | 2864 | 2871 | 2877 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| ·46 | 2884 | 2891 | 2897 | 2904 | 2911 | 2917 | 2924 | 2931 | 2938 | 2944 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| ·47 | 2951 | 2958 | 2965 | 2972 | 2979 | 2985 | 2992 | 2999 | 3006 | 3013 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| ·48 | 3020 | 3027 | 3034 | 3041 | 3048 | 3055 | 3062 | 3069 | 3076 | 3083 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| ·49 | 3090 | 3097 | 3105 | 3112 | 3119 | 3126 | 3133 | 3141 | 3148 | 3155 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |

## ANTILOGARITHMS

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·50 | 3162 | 3170 | 3177 | 3184 | 3192 | 3199 | 3206 | 3214 | 3221 | 3228 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| ·51 | 3236 | 3243 | 3251 | 3258 | 3266 | 3273 | 3281 | 3289 | 3296 | 3304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| ·52 | 3311 | 3319 | 3327 | 3334 | 3342 | 3350 | 3357 | 3365 | 3373 | 3381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| ·53 | 3388 | 3396 | 3404 | 3412 | 3420 | 3428 | 3436 | 3443 | 3451 | 3459 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| ·54 | 3467 | 3475 | 3483 | 3491 | 3499 | 3508 | 3516 | 3524 | 3532 | 3540 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| ·55 | 3548 | 3556 | 3565 | 3573 | 3581 | 3589 | 3597 | 3606 | 3614 | 3622 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| ·56 | 3631 | 3639 | 3648 | 3656 | 3664 | 3673 | 3681 | 3690 | 3698 | 3707 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| ·57 | 3715 | 3724 | 3733 | 3741 | 3750 | 3758 | 3767 | 3776 | 3784 | 3793 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| ·58 | 3802 | 3811 | 3819 | 3828 | 3837 | 3846 | 3855 | 3864 | 3873 | 3882 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| ·59 | 3890 | 3899 | 3908 | 3917 | 3926 | 3936 | 3945 | 3954 | 3963 | 3972 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| ·60 | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| ·61 | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·62 | 4169 | 4178 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·63 | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·64 | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·65 | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·66 | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| ·67 | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| ·68 | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| ·69 | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| ·70 | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| ·71 | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| ·72 | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| ·73 | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| ·74 | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5585 | 5598 | 5610 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| ·75 | 5623 | 5636 | 5649 | 5662 | 5675 | 5689 | 5702 | 5715 | 5728 | 5741 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| ·76 | 5754 | 5768 | 5781 | 5794 | 5808 | 5821 | 5834 | 5848 | 5861 | 5875 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| ·77 | 5888 | 5902 | 5916 | 5929 | 5943 | 5957 | 5970 | 5984 | 5998 | 6012 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| ·78 | 6026 | 6039 | 6053 | 6067 | 6081 | 6095 | 6109 | 6124 | 6138 | 6152 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| ·79 | 6166 | 6180 | 6194 | 6209 | 6223 | 6237 | 6252 | 6266 | 6281 | 6295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| ·80 | 6310 | 6324 | 6339 | 6353 | 6368 | 6383 | 6397 | 6412 | 6427 | 6442 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| ·81 | 6457 | 6471 | 6486 | 6501 | 6516 | 6531 | 6546 | 6561 | 6577 | 6592 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| ·82 | 6607 | 6622 | 6637 | 6653 | 6668 | 6683 | 6699 | 6714 | 6730 | 6745 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| ·83 | 6761 | 6776 | 6792 | 6808 | 6823 | 6839 | 6855 | 6871 | 6887 | 6902 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| ·84 | 6918 | 6934 | 6950 | 6966 | 6982 | 6998 | 7015 | 7031 | 7047 | 7063 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 15 |
| ·85 | 7079 | 7096 | 7112 | 7129 | 7145 | 7161 | 7178 | 7194 | 7211 | 7228 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| ·86 | 7244 | 7261 | 7278 | 7295 | 7311 | 7328 | 7345 | 7362 | 7379 | 7396 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| ·87 | 7413 | 7430 | 7447 | 7464 | 7482 | 7499 | 7516 | 7534 | 7551 | 7568 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| ·88 | 7586 | 7603 | 7621 | 7638 | 7656 | 7674 | 7691 | 7709 | 7727 | 7745 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| ·89 | 7762 | 7780 | 7798 | 7816 | 7834 | 7852 | 7870 | 7889 | 7907 | 7925 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| ·90 | 7943 | 7962 | 7980 | 7998 | 8017 | 8035 | 8054 | 8072 | 8091 | 8110 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| ·91 | 8128 | 8147 | 8166 | 8185 | 8204 | 8222 | 8241 | 8260 | 8279 | 8299 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| ·92 | 8318 | 8337 | 8356 | 8375 | 8395 | 8414 | 8433 | 8453 | 8472 | 8492 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| ·93 | 8511 | 8531 | 8551 | 8570 | 8590 | 8610 | 8630 | 8650 | 8670 | 8690 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| ·94 | 8710 | 8730 | 8750 | 8770 | 8790 | 8810 | 8831 | 8851 | 8872 | 8892 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| ·95 | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 17 | 19 |
| ·96 | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| ·97 | 9333 | 9354 | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| ·98 | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| ·99 | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 |

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